The ultrafilter monad

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11+11=22

Monads

Definition

A **monad** on C is an endofunctor T, a **unit** natural transformation $\eta : id \Rightarrow T$ and a **multiplication** natural transformation $\mu : T^2 \Rightarrow T$ that is unital and associative, i.e. so the following diagrams commute.

$$\begin{tikzcd}
T \arrow[r, "\eta T"] \arrow[dr, "id"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "id"] \\
 & T &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"] & T
\end{tikzcd}$$

Here $T\eta$ and ηT are defined componentwise by **whiskering**: $(T\eta)_c = T(\eta_c)$ and $(\eta T)_c = \eta_{T(c)}$.

Monads from adjunctions

Definition

An **adjunction**

$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & D \arrow[l, bend left, "U"] \end{tikzcd}$$

is a pair of functors and a pair of natural transformations $\eta : 1_C \to UF$ and $\epsilon : FU \to 1_D$, called the **unit** and **counit** respectively, satisfying the triangle identities:

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "id"'] & FUF \arrow[d, "\epsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "\eta U"] \arrow[dr, "id"'] & UFU \arrow[d, "U\epsilon"] \\ & U \end{tikzcd}$$

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Lemma

An adjunction gives rise to a monad $T = UF : C \to C$.

Monads from adjunctions

Example

There is an adjunction

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "\mathbb{Z}[-]"] \arrow[r, phantom, "\perp"] & \mathbf{Ab} \arrow[l, bend left, "U"] \end{tikzcd}$$

where U is the forgetful functor, and $\mathbb{Z}[X]$ is the **free abelian group** generated by X.

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where U is the forgetful functor, and $\mathbb{Z}[X]$ is the **free abelian group** generated by X.
The corresponding monad $\mathbb{Z}[X] : \mathbf{Set} \to \mathbf{Set}$ maps X to the set of finite formal sums $\sum_i a_i x_i$. The unit $\eta_X : X \to \mathbb{Z}[X]$ maps x to the singleton sum $\sum^1 x$. Multiplication is given by distributing coefficients:

$$\mu_X : \mathbb{Z}[\mathbb{Z}[X]] \to \mathbb{Z}[X], \quad \sum_i a_i \sum_j b_j x_j \mapsto \sum_{i,j} a_i b_j x_j$$

Adjunctions from monads

Given a monad (T, η, μ) on C, can we find an adjunction
$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & D \arrow[l, bend left, "U"] \end{tikzcd}$$
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Lemma

Yes. Furthermore, there is both a final and an initial such adjunction. The final adjunction goes to the **category of T-algebras**. *The initial adjunction goes to the* **category of free T-algebras**.

The category of T-algebras

Intiutively, the category consists of **evaluation maps** $a : TA \to A$ where $A \in C$. These are required to play nicely with the unit and multiplication. Maps are maps $f : A \to B$ in C such that the following diagram commutes:

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ A & \xrightarrow{f} & B \end{array}$$

The category of T-algebras

Example

Recall the monad $T = \mathbb{Z}[-] : \mathbf{Set} \to \mathbf{Set}$. **Claim**: the category of T-algebras is equivalent to **Ab**. A $T-$algebra is a set X and an evaluation map $a : \mathbb{Z}[X] \to X$ carrying a formal sum to its corresponding element. The commutative square

$$\begin{tikzcd}
\mathbb{Z}[\mathbb{Z}[X]] \arrow[r, "\mu_X"] \arrow[d, "\mathbb{Z}[a]"'] & \mathbb{Z}[X] \arrow[d, "a"] \\
\mathbb{Z}[X] \arrow[r, "a"'] & A
\end{tikzcd}$$

gives the group axioms.

Monadic adjunctions

When an adjunction is equivalent to the adjunction to the category of $T-$algebras, the adjunction is called **monadic.** The familiar free-forgetful adjunctions between **Set** and **Grp, Ab**, **Ring**, $\mathbf{Vect_k}$ are all monadic, identifying their categories as **algebraic theories**.

Generators and relations

Algebraic theories as we know them can be explained in terms of generators and relations. Given an abelian group A, a set of generators G and a set of relations R, $A = \mathbb{Z}[G]/\mathbb{Z}[R]$. Equivalently, A is the coequalizer

$$\mathbb{Z}[R] \underset{0}{\overset{ev}{\rightrightarrows}} \mathbb{Z}[G] \longrightarrow A$$

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This is not well-behaved categorically, as it is not functorial. Instead, we may take *every element* of A to be a generator, and *every* possible equation to be a relation. This identifies A as the coequalizer

$$\mathbb{Z}[\mathbb{Z}[A]] \underset{\mu_A}{\overset{\mathbb{Z}[a]}{\rightrightarrows}} \mathbb{Z}[A] \xrightarrow{a} A$$

Generators and relations (2)

Theorem

In the category of $T-$algebras for a monad T, an element A is the coequalizer

$$T^2A \underset{\mu_A}{\overset{T[a]}{\rightrightarrows}} TA \xrightarrow{a} A$$

We will see a surprising consequence...

An unrelated story?

There is an adjunction

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "\beta"] \arrow[r, phantom, "\perp"] & \mathbf{CHaus} \arrow[l, bend left, "U"] \end{tikzcd}$$

between sets and compact Hausdorff spaces. β is given by **Stone–Čech compactification** on the set with the discrete topology. The induced monad has an explicit description as the set $\beta(X)$ of **ultrafilters** on X.

Ultrafilters

Definition

An **ultrafilter** $\mathcal{F}$ on a set X is a set of subsets such that whenever we write $X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n$ as a finite disjoint union, exactly one of the X_i's is in $\mathcal{F}$.

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Example

Given any $x \in X$ its **principal ultrafilter** $\mathcal{F}_x$ is the set of all subsets containing x. Note given any partition $X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n$, exactly one X_i contains x.

Limits of ultrafilters

Ultrafilters classify convergence on a topological space.

Definition

$x \in X$ is a **limit** of an ultrafilter $\mathcal{F}$ if every open neighbourhood of x is contained in $\mathcal{F}$.

For example, x is a limit of the principal ultrafilter $\mathcal{F}_x$.

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Lemma

If a topological space X is compact, then every ultrafilter on X has at least one limit. If X is Hausdorff, then every ultrafilter on X has at most one limit. Therefore, X **is compact Hausdorff if and only if every ultrafilter has exactly one limit**.

Algebras over the ultrafilter monad

A $\beta-$algebra is a map $\lim : \beta(X) \to X$ sending each ultrafilter $\mathcal{F}$ to a unique $\lim \mathcal{F} \in X$. By the previous lemma, this hints that $\beta-$algebras are exactly compact Hausdorff spaces. And indeed...

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Lemma

CHaus *is equivalent to the category of* β*-algebras, i.e. the adjunction* $\beta \dashv U$ *is monadic.*

This means compact Hausdorff spaces are an algebraic theory!

Generators and relations on compact Hausdorff spaces

A compact Hausdorff space X is the coequaliser

$$\beta^2 X \underset{\mu_X}{\overset{\beta[\lim]}{\rightrightarrows}} \beta X \xrightarrow{\lim} X$$

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For example of a relation, fix an ultrafilter $\mathcal{F}$ on X and take the principal ultrafilter of ultrafilters $\mathcal{G}_{\mathcal{F}} \in \beta^2 X$. The multiplication takes an ultrafilter of ultrafilters $\mathcal{G}$ to the ultrafilter $\{A \in X : [A] \in \mathcal{G}\}$ where $[A]$ is the set of all ultrafilters containing A. In particular, $\mu_X(\mathcal{G}_{\mathcal{F}}) = \mathcal{F}$.

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This relation therefore says

$$\lim \mathcal{G}_{\lim \mathcal{F}} = \lim \mathcal{F}$$

as expected.

Thank you for listening!

References

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nLab authors, *ultrafilter*, `https://ncatlab.org/nlab/show/ultrafilter`, 2022, Revision 43.

Emily Riehl, *Category theory in context*, Dover Publications, 2017.

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The category of T-algebras

Definition

The category $\mathbf{C^T}$ of $T-\textit{algebras}$ has as objects pairs $(A \in C, a : TA \to A)$ such that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "id"'] & TA \arrow[d, "a"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "\mu_A"] \arrow[d, "Ta"'] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & A \end{tikzcd}$$

Maps are maps $f : A \to B$ in C such that the following diagram commutes:

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"] & TB \arrow[d, "b"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Topology of β

$\beta(X) \in$ **CHaus** has a topology generated by the open sets

$$\mathcal{F}^A := \{\mathcal{F} \in \beta(X) : A \in \mathcal{F}\} \text{ where } A \subset X.$$

β-homomorphisms

A $\beta-$homomorphism is a map $X \to Y$ such that the following square commutes

$$\begin{array}{ccc} \beta(X) & \xrightarrow{\beta f} & \beta(Y) \\ {\scriptstyle \lim}\downarrow & & \downarrow{\scriptstyle \lim} \\ X & \xrightarrow{f} & Y \end{array}$$

where $\beta f(\mathcal{F})$ is the ultrafilter that takes a partition $Y = Y_1 \sqcup \cdots \sqcup Y_n$, and picks out the unique Y_i such that $f^{-1}(Y_i) \in \mathcal{F}$.