The pre-monoidal envelope of an ∞-operad

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Overview

Pre-monoidal ∞-operads were introduced in [KSW24], under the name pre-coCartesian ∞-operads, to describe a class of operads that appears in the study of constructible factorisation algebras. In that setting, one studies algebras over operads $\mathsf{Disks}^{\sqcup}_{/\mathcal{M}}$ of disjoint unions of disks, or multidisks, embedded into a suitable stratified manifold $\mathcal{M}$. For instance, constructible factorisation algebras on $\mathbb{R}^n$ with its trivial stratification are simply $\mathbb{E}_n$-algebras.

While $\mathsf{Disks}^{\sqcup}_{/\mathcal{M}}$ is not even partially symmetric monoidal, it does have the following structure: every joint embedding $\alpha : (U_i)_i \to V$ over $\mathcal{M}$ factors uniquely through an embedding of $\sqcup U_i$ into $\mathcal{M}$. This is the structure of a **pre-monoidal ∞-operad**. KSW show it is relatively easy to check if a map of pre-monoidal ∞-operads is a localisation [KSW24][A.4.7], and use this to prove that two models of constructible factorisation algebras are equivalent.

We aim to answer the following natural question about pre-monoidal $\infty-$operads: do the natural functors $\mathsf{smCat}_\infty \xrightarrow{U_1} \mathsf{pMonOp}_\infty \xrightarrow{U_2} \mathsf{Op}_\infty$ have left adjoints?

Background on ∞-operads

We let $\mathbb{F}_*$ be the category of finite pointed sets, and say a function is **active** if it only sends the basepoint to the basepoint. An ∞-operad encodes the structure of a **multicategory**. Namely, it has a space $\mathcal{O}^{\simeq}$ of objects, and for every n-tuple $\underline{x} \in (\mathcal{O}^{\simeq})^n$ and object $y \in \mathcal{O}^{\simeq}$, spaces $\mathsf{Mul}(\underline{x}, y)$ of **multimorphisms**, as well as a **multicomposition** rule. ∞-operads can be modelled as certain functors $\pi : \mathcal{O}^{\otimes} \to \mathbb{F}_*$ [Lur17][Def. 2.1.1.10], where in particular we require that $\mathcal{O}^{\otimes}_{\langle n\rangle} \simeq (\mathcal{O}^{\otimes}_{\langle 1\rangle})^n$. We can then identify the space $\mathsf{Mul}_{\mathcal{O}^{\otimes}}(\underline{x}, y)$ of multimorphisms with the fiber $\mathcal{O}^{\otimes}_a(\underline{x}, y)$ over the unique active map $a : \pi(\underline{x}) \to \langle 1\rangle$. Given a symmetric monoidal ∞-category $C^{\otimes}$, we can define an ∞-category $\mathsf{Alg}_{\mathcal{O}^{\otimes}}(C^{\otimes})$ of **$\mathcal{O}^{\otimes}$-algebras in $C^{\otimes}$**. For instance, there is an operad $\mathbb{E}_\infty$ whose algebras in $C^{\otimes}$ are commutative monoids.

Pre-monoidal $\infty-$operads

Definition. An ∞-operad $\pi : \mathcal{O}^{\otimes} \to \mathbb{F}_*$ is **pre-monoidal** if there is a class pMon of **pre-monoidal** morphisms, such that $(\mathsf{pMon}, \pi\text{-}\mathsf{inv}))$ is an orthogonal factorisation system on $\mathcal{O}^{\otimes}_{act}$. Let pMonOp_∞ be the ∞-category of pre-monoidal ∞-operads and operad maps that preserve pre-monoidal morphisms.

Example ([KSW24] A.4.6, [AFT16] 1.20)**.** Given a symmetric monoidal category $C^{\otimes}$ whose unit is initial, a symmetric monoidal subcategory $D^{\otimes} \subset C^{\otimes}$, and an object $x \in C$, we can form a pre-monoidal operad $(D_{/x})^{\otimes}$ whose objects lie in $D_{/x}$ and where a multimorphism $(f_i : y_i \to x)_{i \in I} \to (z \to x)$ is a map $(f : \otimes y_i \to x)$ extending the f_i's, and a map $\otimes y_i \to z$ over x. The operad $\mathsf{Disks}^{\sqcup}_{/\mathcal{M}}$ is of this form.

Main results

Let $\mathcal{O}^{\otimes}$ be a pre-monoidal ∞-operad, $\mathsf{Env}(O)^{\otimes}$ the *symmetric monoidal envelope* of [Lur17][Constr. 2.2.4.1], and consider the wide subcategory $\mathcal{W} \subset \mathsf{Env}(\mathcal{O})^{\otimes}$ of its monoidal envelope generated under tensor product by the pre-monoidal morphisms.

Theorem. *There is a left adjoint E_1 to U_1 whose value on a pre-monoidal operad $\mathcal{O}^{\otimes}$ is the localisation $\mathsf{E}_1(\mathcal{O}^{\otimes}) \simeq \mathsf{Env}(\mathcal{O}^{\otimes})[\mathcal{W}^{-1}]$.*

Let $\pi : \mathcal{O}^{\otimes} \to \mathbb{F}_*$ be an ∞-operad, and $\mathsf{Act}(\mathcal{O}^{\otimes}) \subset \mathsf{Ar}(\mathcal{O}^{\otimes})$ the full subcategory spanned by the active morphisms, equipped with the functor $\pi\mathsf{t}$ to $\mathbb{F}_*$, where t is the target. Let $\mathcal{J} \subset \mathsf{Act}(\mathcal{O}^{\otimes})_{act}$ be the wide subcategory $\mathsf{s}\text{-}\mathsf{inv} \cap \pi\mathsf{t}\text{-}\mathsf{inv}$, where s is the source.

Theorem. *There is a left adjoint $\mathsf{pEnv} : \mathsf{Op}_\infty \to \mathsf{pMonOp}_\infty$, called the **pre-monoidal envelope**, which is left adjoint to U_2 and given on operads $\mathcal{O}^{\otimes}$ by $\mathsf{pEnv}(\mathcal{O}^{\otimes}) \simeq \mathsf{Act}(\mathcal{O}^{\otimes})[\mathcal{J}^{-1}]$.*

[AFT16] David Ayala, John Francis, and Hiro Lee Tanaka, *Factorization homology of stratified spaces*, 2016.

[KSW24] Eilind Karlsson, Claudia I Scheimbauer, and Tashi Walde, *Assembly of constructible factorization algebras.*

[Lur17] Jacob Lurie, *Higher algebra*, 2017.