Eisenboaties 07/04/2023

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4.1 The Structure Sheaf

$$D(f) = \{ [p] \in \operatorname{Spec} A \mid f \notin p \}$$

Def 4.1.1: $\mathcal{O}_{\operatorname{Spec} A}(D(f))$ is the localization of A at the multiplicative subset of elements that do not vanish outside of $V(f)$.

$(g \in A \text{ s.t. } V(g) \subset V(f))$

Ex 4.1.2: $A_f \to \mathcal{O}_{\operatorname{Spec} A}(D(f))$ is iso.

$(D(f) \subset D(g))$ $\Updownarrow$ g is invertible in A_f

Thm 4.1.2: This defines a sheaf on $\operatorname{Spec} A$

Pf: 1) Base identity, $\operatorname{Spec} A = \bigcup_{i \in I} D(f_i)$

$$0 \to A \to \prod_{i \in I} A_{f_i} \to \prod_{i,j \in I} A_{f_i f_j}$$

is exact.

2) Gluing: $\frac{a_i}{g_i} \in A_{f_i}$

$$(g_i g_j)^m (g_j a_i - g_i a_i) = 0$$

$$h_j b_i = h_i b_j \qquad h_i = g_i^{m+1}, \quad b_i = a_i g_i^m$$

$$\bigcup D(h_i) = \operatorname{Spec} A \iff \exists r_i \in A \text{ s.t. } \sum_{i \in I} r_i h_i = 1$$

$$\bigcup_{i\in I} D(h_i) = \operatorname{Spec} A \iff \exists\, r_i \in A \text{ s.t. } \sum_{i\in I} r_i h_i = 1$$

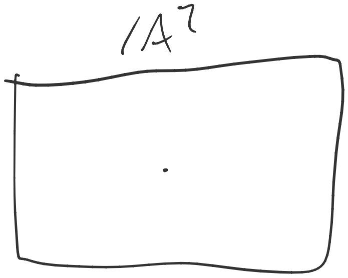
$$r = \sum r_i b_i$$

$$r h_j = b_j \qquad r|_{A_j} = \frac{r_i}{g_i}$$

M an A-module,

$$\tilde{M}(D(g)) = M_g = \left\{ \frac{m}{g^n} \;\middle|\; n \in \mathbb{N},\ m \in M \right\}$$

$\rightsquigarrow \tilde{M}$ is an $\mathcal{O}_{\operatorname{Spec} A}$-module



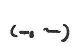
$$\mathbb{A}^2 - (0,0) \rightsquigarrow k[x,y]$$

4.2 Visualizing schemes

$\mathbb{C} \rightsquigarrow$

$\mathbb{C}[x]/(x^2) \rightsquigarrow$ 

$\mathbb{C}[x]/(x^3)$ 

$\mathbb{C}[x,y]/(x^2, y)$ 

$\mathbb{C}[x,y]/(x^2, y^2)$ 

$\mathbb{C}[x,y]/(x,y)^2$ 

$\mathbb{C}[x, y] / (x, y)^2$

4.3 Definition of schemes

Def 4.3.1: · An <u>isomorphism of ringed spaces</u> $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is

i) $\pi : X \to Y$ homeomorphism

ii) $\pi^\# : \mathcal{O}_Y \to \pi_* \mathcal{O}_X$ isomorphism

· An <u>affine scheme</u> is a ringed space iso to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$

· A <u>scheme</u> $(X, \mathcal{O}_X)$ is a ringed space s.t every point of X has a nbhd U s.t $(U, \mathcal{O}_X|_U)$ is affine.

Ex 4.3.A: Describe a bijection between

$$\{\operatorname{Spec} A \xrightarrow{\sim} \operatorname{Spec} A'\} \leftrightarrow \{\text{ring iso } A' \to A\}$$

Hint: $\pi : \operatorname{Spec} A \to \operatorname{Spec} A'$ induces $\pi^\# : A' \to A$ induces

$\rho : \operatorname{Spec} A \to \operatorname{Spec} A'$

Show $\pi = \rho$ on i) points

ii) top logy

iii) sheaves

<u>**Ex 4.3.B:**</u> For $f \in A$, show that

$$\left(D(f), \mathcal{O}_{\operatorname{Spec} A}\big|_{D(f)}\right) \cong \left(\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f}\right)$$

Def 4.3.2: If $U \subset X$ open, $(U, \mathcal{O}_X|_U)$ is an <u>open subscheme</u>.

If it is affine, we call it an <u>affine open</u>.

Ex 4.3.F: The stalk of $\mathcal{O}_{\operatorname{Spec} A}$ at $[p]$ is the local ring A_p

Def 4.3.6: A ringed space is locally ringed if stalks are local rings.

$\rightsquigarrow$ quot by maximal ideal in stalks

$\rightsquigarrow$ $\mathcal{O}_{X,p} / m_p = \kappa(p)$ residue field

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$27/4$ function on $\operatorname{Spec} \mathbb{Z} - [(2)]$

- value at 5 : -2 $\quad \mathbb{F}_5$
- value at (0) : $\frac{27}{4}$ $\quad \mathbb{Q}$
- value at 3 : 0 $\quad \mathbb{F}_3$
- value at 7 : -2 $\quad \mathbb{F}_7$

4.4 More Examples

1) $\mathbb{A}^2_k - \{(0,0)\}$ $\rightsquigarrow$ is not distinguished open

$$\mathbb{A}^2_k - \{(0,0)\} = D(x) \cup D(y)$$

$\rightsquigarrow$ functions on $\underbrace{D(x)}_{A_x}$ and $\underbrace{D(y)}_{A_y}$ agreeing on $\underbrace{D(xy)}_{A_{xy}}$

$$A_x \cap A_y = k[x,y]$$

If it is affine, $(U, \mathcal{O}_U) \cong (\operatorname{Spec} k[x,y], \mathcal{O}_{\operatorname{Spec} k[x,y]})$

ideal (x,y) should cut out something nonempty ↯

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Ex 4.4.A: Given

- schemes X_i
- open subschemes $X_{ij} \subset X_i$ w/ $X_{ii} = X_i$
- isomorphisms $f_{ij} : X_{ij} \to X_{ji}$ w/ $f_{ii} = id$

s.t.

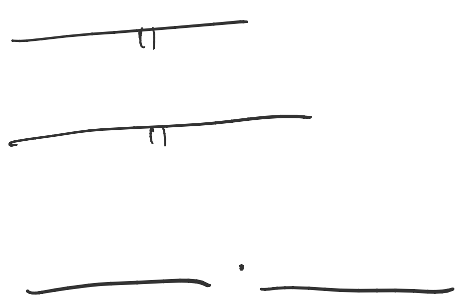
$$f_{ik} = f_{jk} \circ f_{ij}$$

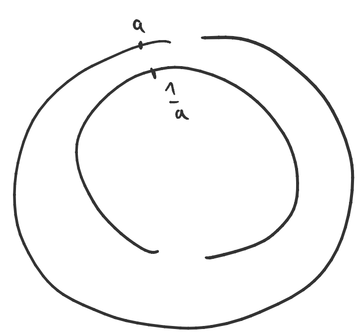
There is a unique up to unique iso. scheme X w/ open subschemes iso to X_i compatible to this gluing data.

2) $\mathbb{A}^1 - \{0\} = \operatorname{Spec} k[t, 1/t] = \operatorname{Spec} k[u, \frac{1}{u}]$

3) $t \to u$ → gluing $\mathbb{A}^1$ along this

$t \to \frac{1}{u}$





→ projective line over k $\mathbb{P}^1_k$

4.3 Projective schemes + Proj construction

Def 4.3.3: An $\mathbb{Z}$-graded ring is $S_\bullet = \bigoplus_{n \in \mathbb{Z}} S_n$ s.t.

$$S_m S_n \subset S_{m+n}$$

- S_0 is a ring
- S_n is a S_0-module
- $S_\bullet$ is a S_0-algebra

. Elements in S_n are <u>homogeneous</u>, of <u>degree</u> n

. An ideal $I \trianglelefteq S_\bullet$ is <u>homogeneous</u> if it is gen. by hom. elements.

<u>Ex 4.5.C</u>: a) $I = \bigoplus_{n \in \mathbb{Z}} I_n$ and $S_\bullet / I$ has a $\mathbb{Z}$-grading

b) hom. is compatible w/ sum, product, radical

c) $I \subset S_\bullet$ is prime if for any homogeneous $a, b \in S_\bullet$
$ab \in I \Rightarrow a \in I$ or $b \in I$.

d) If T is a mult. subset of hom. elements, $T^{-1} S_\bullet$ has a $\mathbb{Z}$-grading

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If $S_0 = A$, we call $S_\bullet$ <u>graded over A</u>, $S_+ = \bigoplus_{i > 0} S_i$ <u>irrelevant ideal</u>

if S_+ is fin. gen. over A, we call $S_\bullet$ <u>fin. gen. graded ring</u> over A. If $S_\bullet$ is generated by S_1 as an A-algebra, $S_\bullet$ is <u>gen. in degree 1</u>.

Proj $S_\bullet$

= built by gluing affine pieces

$k[x_0, x_1, \dots, x_n]$

$k\left[\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}\right]$

for $f \in S_+$ homogeneous, consider

$$\operatorname{Spec}\left(\left((S_\bullet)_f\right)_0\right)$$

· points = hom prime ideals

Ex 4.5.E: a) bijection between prime ideals of $((S_\bullet)_f)_0$ and hom. prime ideals of $(S_\bullet)_f$

b) set of prime ideals of $((S_\bullet)_f)_0$ as a subset of Proj $S_\bullet$

If T is a set of hom. elements, define $V(T) \subset \operatorname{Proj} S_\bullet$

Ex 4.5.F: $D(f)$ "is" $\operatorname{Spec}((S_\bullet)_f)_0$

→ topology on Proj $S_\bullet$

$$D(f) \hookrightarrow \operatorname{Proj} S_\bullet$$

Ex 4.5.K: If $f, g \in S_+$

$$\operatorname{Spec}((S_\bullet)_{fg})_0 \xrightarrow{\sim} D(g^{\deg f}/f^{\deg g}) \subset \operatorname{Spec}((S_\bullet)_f)_0$$

→ agree on triple overlaps

→ define a sheaf $\mathcal{O}_{\operatorname{Proj} S_\bullet}$

Def 4.5.8: $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n]$

Ex 4.5.P: For $f \in S_+$ homog, define $V(f)$ "in" Proj $S_\bullet$. the vanishing scheme

Def 4.5.9: A scheme of the form Proj $S_\bullet$ is a projective A-scheme

A quasiprojective scheme is an open subscheme of "

V a k-vector space, define a ring

$$\mathrm{Sym}^\bullet V^\vee = k \oplus V^\vee \oplus \mathrm{Sym}^2 V^\vee \oplus \cdots$$

$$\mathrm{Sym}^2 V^\vee = V^\vee \otimes V^\vee / \langle a \otimes b - b \otimes a \mid a, b \in V^\vee \rangle$$

$$\mathbb{P}V := \mathrm{Proj}(\mathrm{Sym}^\bullet V^\vee)$$