More on Simplicial Cats

Recall from last week:

- Monoidal categories, functors, nat. transformations, etc.
- Enriched categories, functors, nat. transformations, etc.

Lemma. We have a 2-functor

$$\underline{\mathrm{MonCat}} \longrightarrow \underline{\mathrm{Cat}}$$

$$\Phi : V \longrightarrow V' \rightsquigarrow \Phi_* : \underline{\mathrm{Cat}}_V \longrightarrow \underline{\mathrm{Cat}}_{V'} \quad \text{a 1-functor}$$

$$\mathrm{ob}\, \Phi_* \mathcal{C} = \mathrm{ob}\, \mathcal{C}, \qquad \mathrm{Hom}_{\Phi_* \mathcal{C}}(x,y) = \Phi \mathrm{Hom}_{\mathcal{C}}(x,y).$$

This sends monoidal adjunctions to adjunctions.

Def: $\underline{\mathrm{Cat}}_\Delta$ is the category of simplicial categories, ie. categories enriched over sSet.

Lemma: $\underline{\mathrm{Cat}}_\Delta$ is bicomplete.

Recall from earlier: we have adjunctions

π_0 – connected components

$$\begin{tikzcd} \underline{\mathrm{sSet}} \arrow[r, bend left=60, "\pi_0"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, bend right=60, "\mathrm{ev}_0"'] & \underline{\mathrm{Set}} \arrow[l, "c"'] \end{tikzcd}$$

c – constant simplicial set

ev_0 – evaluation at $[0]$

Lemma. These all of these functors are monoidal, and so are the adjunctions, so we get adjunctions

$$\begin{tikzcd}[column sep=huge]
\underline{\mathrm{Cat}}_\Delta \arrow[r, bend left=50, "\tau = (\pi_0)_*"] \arrow[r, bend left=25, phantom, "\perp"] \arrow[r, bend right=50, "\underline{u} = (\mathrm{ev}_0)_*"'] \arrow[r, bend right=25, phantom, "\perp"'] & \underline{\mathrm{Cat}} \arrow[l, "c = c_*"']
\end{tikzcd}$$

$\tau = (\pi_0)_*$ – homotopy category

$c = c_*$ – constant enrichment

$\underline{u} = (\mathrm{ev}_0)_*$ – underlying category

$\underline{\mathrm{Hom}(\Delta^0, -)_*}$

Idea of proof: both $\underline{\mathrm{sSet}}$ and $\underline{\mathrm{Set}}$ are cartesian ($\otimes = \times$), so we have canonical maps

$$\begin{tikzcd}[row sep=small]
 & X \\
X \times Y \arrow[ur] \arrow[dr] & \\
 & Y
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}[row sep=small]
 & FX \\
F(X \times Y) \arrow[ur] \arrow[dr] & \\
 & FY
\end{tikzcd}
\rightsquigarrow F(X \times Y) \to FX \times FY.$$

- c and ev_0 are right adjoints, so they preserve products.
- For π_0, consider

$$\begin{tikzcd}
(X \times Y)_0 \arrow[r, "\cong"] \arrow[d] & X_0 \times Y_0 \arrow[d] \\
\pi_0(X \times Y) \arrow[r] & \pi_0 X \times \pi_0 Y
\end{tikzcd}$$

must be surjective

also injective since we can lift relations, because ev_1 is monoidal. $\square$

Terminology. Let $\mathcal{C} \in \underline{\mathrm{Cat}}_\Delta$, a morphism $x \to y$ in $\mathcal{C}$ is a morphism $x \to y$ in $u\mathcal{C}$, i.e. a 0-simplex of $\mathrm{Hom}_{\mathcal{C}}(x, y)$. A morphism in $\mathcal{C}$ is an equivalence if its image in $\tau\mathcal{C}$ is an isomorphism.

Next: defining the nerve of a simplicial category.

We do this by constructing a cosimplicial object in $\underline{\mathrm{Cat}_\Delta}$.

$\Delta \to \underline{\mathrm{Cat}_\Delta}$

Def. Let J be a finite non-empty totally ordered set, $i,j \in J$.

$$P_{i,j} = \{ I \subseteq J : i,j \in I \text{ and } k \in I \Rightarrow i \leq k \leq j \}$$
$$= \{ \text{subsets of } [i,j] \text{ containing the endpoints} \}.$$

$P_{i,j}$ is a poset under inclusion.

If $i > j$, then $P_{i,j} = \emptyset$. If $i \leq j \leq k \in J$, then

$$N(P_{i,j}) \times N(P_{j,k}) \longrightarrow N(P_{i,k})$$
$$(I, I') \longmapsto I \cup I'$$

is an associative binary operation. $[n]$

For each J, we have a simplicial category $\mathbb{C}[\Delta^J]$:

$$\mathrm{ob}\, \mathbb{C}[\Delta^J] = J, \qquad \mathrm{Hom}_{\mathbb{C}[\Delta^J]}(i,j) = \begin{cases} \emptyset & \text{if } i > j \\ N(P_{i,j}) & \text{o/w.} \end{cases}$$

Composition is induced by the binary operation above.

Lemma. Let $n \geq 1$, $J = [n]$. Then

(a) $N(P_{0,n}) \cong (\Delta^1)^{n-1}$

(b) $P_{i,j} \cong P_{0,j-i}$.

Proof. (b) is clear. For (a), suffices to show

$$P_{0,n} \cong \underbrace{[1] \times \cdots \times [1]}_{n-1 \text{ times}}$$

an element of this represents a subset of $(0,n)$ in $[n]$.

$$S \longmapsto (\chi_S(1), \ldots, \chi_S(n-1))$$

Lemma. If $\mathcal{C}$ is a category with an initial or terminal object, then $N(\mathcal{C})$ is contractible.

$\emptyset \in \mathcal{C}$ initial $\Rightarrow$ $\emptyset \Rightarrow 1_{\mathcal{C}}$

$\mathcal{C} \times [1] \longrightarrow \mathcal{C} \rightsquigarrow N(\mathcal{C}) \times \Delta^1 \longrightarrow N(\mathcal{C})$

Cor. For $i \leq j$, $\mathrm{Hom}_{\mathfrak{C}[\Delta^J]}(i,j)$ is contractible.

Lemma. There is a unique isomorphism $\pi\mathfrak{C}[\Delta^n] \cong [n]$ which is the identity on objects. Hence, we have a canonical functor $\mathfrak{C}[\Delta^n] \longrightarrow c[n]$.

$\mathfrak{C}[\Delta^-]$

Lemma. The assignment $J \longmapsto \mathfrak{C}[\Delta^J]$ defines a functor

$$\underline{\mathrm{LinOrdSet}} \longrightarrow \underline{\mathrm{Cat}}_\Delta .$$

In particular, we have a cosimplicial object $\Delta \to \underline{\mathrm{Cat}}_\Delta$ given by $[n] \to \mathfrak{C}[\Delta^n]$.

Idea of proof: Unfolding definitions, it suffices to construct for any monotone map $f: J \to J'$ and $i \leq j \in J$, a monotone map $P_{i,j} \longrightarrow P_{f(i),f(j)}$. This is $I \mapsto f(I)$. $\square$

Def. For $\mathcal{C} \in \underline{\mathrm{Cat}}_\Delta$, its simplicial nerve (or homotopy-coherent nerve) is the sSet:

$$N(\mathcal{C})_n = \mathrm{Hom}_{\underline{\mathrm{Cat}}_\Delta}(\mathfrak{C}[\Delta^n], \mathcal{C})$$

Lemma. If $\mathcal{C} \in \underline{\mathrm{Cat}}$, then $N(\mathcal{C}) \cong N(c\mathcal{C})$.

Proof:

$$\begin{aligned} N(c\mathcal{C})_n &= \mathrm{Hom}_{\underline{\mathrm{Cat}}_\Delta}(\mathfrak{C}[\Delta^n], c\mathcal{C}) \\ &\cong \mathrm{Hom}_{\underline{\mathrm{Cat}}}(\pi\mathfrak{C}[\Delta^n], \mathcal{C}) \\ &\cong \mathrm{Hom}_{\underline{\mathrm{Cat}}}([n], \mathcal{C}) = N(\mathcal{C})_n . \end{aligned}$$

Discussion. What do (low dimensional) simplices in $N(\mathcal{C})$ look like?

$[0,0]$
$\cup\!\!|$

- $\mathbb{C}[\Delta^0]$ has a single object and $\mathrm{Hom}_{\mathbb{C}[\Delta^0]}(0,0) = N(P_{0,0}) = \underline{\Delta^0}$.

$[1]$

- $\mathbb{C}[\Delta^1]$ has two objects and all hom-sSets are Δ^0. $P_{0,1}$

Hence, $\mathbb{C}[\Delta^0] = c[0]$ and $\mathbb{C}[\Delta^1] = c[1]$.

- $N(\mathcal{C})_0 = \mathrm{Hom}_{\underline{\mathrm{Cat}_\Delta}}(\mathbb{C}[\Delta^0], \mathcal{C}) \cong \mathrm{Hom}_{\underline{\mathrm{Cat}}}([0], u\mathcal{C})$
$$\cong \mathrm{ob}\, u\mathcal{C} = \mathrm{ob}\, \mathcal{C}.$$

- $N(\mathcal{C})_1 = \mathrm{Hom}_{\underline{\mathrm{Cat}_\Delta}}(\mathbb{C}[\Delta^1], \mathcal{C}) \cong \mathrm{Hom}_{\underline{\mathrm{Cat}}}([1], u\mathcal{C})$
$$\cong \mathrm{mor}\, u\mathcal{C} = \mathrm{mor}\, \mathcal{C}.$$

What is $\mathbb{C}[\Delta^2]$? It has three objects. All hom-sSets are Δ^0 except $\mathrm{Hom}_{\mathbb{C}[\Delta^2]}(0,2) = N(P_{0,2}) \cong \Delta^1$. $\{0,2\}$, $\{0,1,2\}$

Hence, a simplicial functor $\mathbb{C}[\Delta^2] \to \mathcal{C}$ picks out:

gf, h
$\in \mathrm{Hom}(X,Z)$

$\mathbb{C}[\Delta^2]$
$$\begin{tikzcd} & 1 \arrow[dr, "\Delta^0"] & \\ 0 \arrow[ur, "\Delta^0"] \arrow[rr, "\Delta^1"'] & & 2 \end{tikzcd} \rightsquigarrow \begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "1\text{-simplex}"'] & & Z \end{tikzcd} \rightsquigarrow \begin{tikzcd} & Y \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow"] & \\ X \arrow[ur, "f"] \arrow[rr, "\exists h"'] & & Z \end{tikzcd}$$

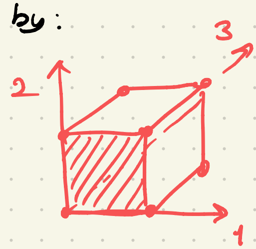
Lemma. There is a unique colimit preserving functor
$$\mathbb{C}[-] : \underline{\mathrm{sSet}} \longrightarrow \underline{\mathrm{Cat}_\Delta}$$
which sends $\Delta^n \mapsto \mathbb{C}[\Delta^n]$. It is left-adjoint to N.

Fact. $\mathbb{C}[-]$ preserves monomorphisms, i.e. if $A \subseteq X$ is an inclusion of sSets then $\mathbb{C}[A] \hookrightarrow \mathbb{C}[X]$.

Lemma. Let $0<j<n$. $\mathfrak{C}[\Lambda^n_j] \subseteq \mathfrak{C}[\Delta^n]$ is given by:

(1) $\mathrm{ob}\,\mathfrak{C}[\Lambda^n_j] = \mathrm{ob}\,\mathfrak{C}[\Delta^n]$.

(2) $\mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(i,k) = \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(i,k)$ except



$$\mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(0,n) \subsetneq \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(0,n)$$

which is given by the subsimplicial set of $(\Delta^1)^{n-1}$ obtained by deleting the interior of the bottom j-face.

Proof idea: $\Lambda^n_j = \bigcup_{i\neq j} \Delta^{n\setminus\{i\}}$ so look at $\mathfrak{C}[\Delta^{n\setminus\{i\}}] \subseteq \mathfrak{C}[\Delta^n]$.

Lemma. $N(\mathcal{C})$ is a composer. Further, if all hom-sSets in $\mathcal{C}$ are Kan complexes then $N(\mathcal{C})$ is an ∞-category.

Proof:

$$\begin{tikzcd} I^n \arrow[r] \arrow[d] & N(\mathcal{C}) \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \mathfrak{C}[I^n] \arrow[r] \arrow[d] & \mathcal{C} \\ \mathfrak{C}[\Delta^n] \arrow[ur, dotted] & \end{tikzcd}$$

But we have a retraction of $\mathfrak{C}[I^n] \to \mathfrak{C}[\Delta^n]$.

To see this, note $\mathfrak{C}[I^n] = c[n]$ since $I^n = I^{n-1} \sqcup_{\Delta^0} I^1$.

But we saw there is a unique simplicial functor $\mathfrak{C}[\Delta^n] \to c[n]$ that is the identity on objects. Then

$$c[n] \cong \mathfrak{C}[I^n] \hookrightarrow \mathfrak{C}[\Delta^n] \to c[n]$$

is the identity.

For Kan complexes, need $0<j<n$

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d] & N(\mathcal{C}) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathfrak{C}[\Lambda^n_j] \arrow[r, "f"] \arrow[d] & \mathcal{C} \\ \mathfrak{C}[\Delta^n] \arrow[ur, dashed] & \end{tikzcd}$$

From the previous lemma, we only need to worry about

$$\begin{tikzcd} \mathrm{Hom}_{\mathfrak{C}[\Lambda^n_j]}(0,n) \arrow[r, "f"] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(f(0), f(1)) \\ \mathrm{Hom}_{\mathfrak{C}[\Delta^n]}(0,n) \arrow[ur, dashed] & \end{tikzcd}$$

But this follows from the fact that the vertical map is $\boxed{\text{anodyne.}}$ One checks that this extension in fact defines a simplicial functor. $\square$

Remark. Nerves of categories enriched over Kan complexes will be $(\infty,1)$-categories?

Def. Consider the simplicial category of CW-complexes and hom-sSets the singular set of the mapping space. Its simplicial nerve is the ∞-category of spaces, $\underline{\mathcal{S}pc}$.

Lemma. The product and coproduct of ∞-cats is an ∞-cat.

Proof. For product, find extension in each factor and then put together. For coproduct, note that Λ^n_j and Δ^n are connected. $\square$

Def. A sub-∞-cat $\mathcal{C}' \subseteq \mathcal{C}$ is a subsimplicial set determined by subsets $X \subseteq \mathcal{C}_0$ and $S \subseteq \mathcal{C}_1$ between objects in X and closed under composition on equivalences. An n-simplex of $\mathcal{C}$ belongs in $\mathcal{C}'$ iff the edges of its restriction to the spine I^n are in S. A subcategory is full if S contains all 1-simplices whose boundary is in X.

Lemma. A sub-∞-category is an ∞-cat. Its homotopy category is the subcategory of $h\mathcal{C}$ on the image of S.

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ N(h\mathcal{C}') \arrow[r] & N(h\mathcal{C}) \end{tikzcd}$$

is a pullback. For any subcategory $\mathcal{D} \subseteq h\mathcal{C}$, this pullback gives a sub-∞-cat.

Cor. There is a 1-to-1 correspondence between sub-∞-cats of $\mathcal{C}$ and subcategories of $h\mathcal{C}$. Full correspond to full.

Def. A natural transformation between functors $f, g : \mathcal{C} \to \mathcal{D}$ is a simplicial map $\mathcal{C} \times \Delta^1 \to \mathcal{D}$ that restricts to f and g.

Remark. This exactly a 1-simplex in $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})$.

We will see that $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})$ is also an ∞-cat and $N(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) = \underline{\mathrm{Hom}}(N\mathcal{C}, N\mathcal{D})$ since N is fully faithful.

$- \times A \dashv \underline{\mathrm{Hom}}(A, -)$

§ Anodyne & (Lan) fibration $\mathcal{C}$ = sSets

Def: A (left/right/inner) fibration $p: K \to S$ has the RLP wrt ~~all~~ left/right/inner horns

right lifting

eg
$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^n \arrow[r] & * \end{tikzcd}$$
is an inner fibration iff X is ∞-cat

$S \subset \mathrm{Mor}(\mathcal{C})$

$\chi_R(S) = \{ f: X \to \mathcal{C} \mid f \text{ have RLP wrt } S \}$ $\chi_L(S)$ (LRP)

$\chi_R(\{\text{left horns}\})$ = Left Fibration.

Def A map $f: A \to B$ is (left/right/inner) anodyne if it has LLP wrt (...) fibrations. i.e. $\chi_L(\chi_R(\text{left/right/inner horns}))$

$S \Rightarrow S \subset \chi_L(\chi_R(S)) =: \chi(S)$

Def: A $S \subset \mathrm{Mor}(\mathcal{C})$ is saturated if:

1) it contains all isos

2) Stable under pushouts:
$$\begin{tikzcd} A \arrow[r] \arrow[d, "S \ni i"'] & A' \arrow[d, "i'"] \\ B \arrow[r] & B' \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$
$\Rightarrow i' \in S$

3) Stable retracts:
$$\begin{tikzcd} A' \arrow[r] \arrow[d, "i'"'] \arrow[rr, bend left, "id"] & A \arrow[r] \arrow[d, "i"] & A' \arrow[d, "i'"] \\ B' \arrow[r] \arrow[rr, bend right, "="'] & B \arrow[r] & B' \end{tikzcd}$$
If $i \in S$, so is i'.

4) Stable under countable composition ($\mathbb{N}$-diagram)
$$\begin{tikzcd} A_0 \arrow[r, "i_0"] \arrow[dr, "i"'] & A_1 \arrow[r, "i_1"] \arrow[d] & A_2 \arrow[r] \arrow[dl] & \cdots \\ & \operatorname{colim}_{\mathbb{N}} A_j & & \end{tikzcd}$$
$i_j \in S \ \forall j$.

$\Rightarrow \in S$

5) Coprods: If $\{A_j \xrightarrow{i_j} B_j\}_{j\in J} \subset S$

$$\Rightarrow \quad \coprod_j A_j \xrightarrow{\coprod i_j} \coprod_j B_j \in S$$

<u>Def:</u> $S \subset Mor(\mathcal{C})$. The <u>saturated closure</u> of S is the smallest saturated set containing S, i.e. $\bar{S} = \bigcap_{\substack{T \text{ saturated} \\ T \supset S}} T$

<u>Lem:</u> $S \subset Mor(\mathcal{C})$. Then, $\chi_L(S)$ is saturated.

<u>Pf:</u>

•
$$\begin{tikzcd} A \arrow[r] \arrow[d, "\phi"', "\sim"] & X \arrow[d, "f\in S"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \quad \checkmark$$

•
$$\begin{tikzcd} A \arrow[r] \arrow[d, "\chi_L(S)\ni"', "i"] & A' \arrow[r] \arrow[d, "i'"] & X \arrow[d, "\in S"] \\ B \arrow[r] \arrow[urr, dashed] & B' \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

•
$$\begin{tikzcd} A' \arrow[r] \arrow[d] \arrow[rr, bend left, "id"] & A \arrow[r] \arrow[d, "i"] & A' \arrow[r] \arrow[d, "i'"] & X \arrow[d, "f\in S"] \\ B' \arrow[r] \arrow[rr, bend right, "id"'] & B \arrow[r] \arrow[urr, dashed] & B' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

•
$$\begin{tikzcd} A_0 \arrow[d, "i_0"] \arrow[rr] \arrow[dr, "i"] & & X \arrow[d, "f\in S"] \\ A_1 \arrow[r] \arrow[d] & \operatorname{colim} A_i \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \\ A_2 \arrow[ur] \arrow[d] & & \\ \vdots & & \end{tikzcd}$$

• similarly for coprods

$\square$

<u>Prop:</u> (Small Object Argument) $\mathcal{C} = sSet$

Let $S = \{A_i \to B_j\} \overset{\text{small}}{\subset} Mor(\mathcal{C})$ s.t. A_i's have finitely many non-deg. simplices

Then any $f: X \to Y$ in sSets can be factorized.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \\ & Z \arrow[ur, "g"'] & \end{tikzcd} \quad \text{s.t.} \quad h \in \bar{S}, \; g \in \chi_R(S).$$