Eisenboaties, Chapter (Pieces of) 8:

USEFUL CLASSES OF MORPHISMS OF SCHEMES

8 : USEFUL MORPHISMS

Throughout $\pi : X \to Y$ morphism of schemes

<u>SLOGAN</u>: morphisms are more fundamental than objects

e.g. P : property of schemes

$\rightsquigarrow$ $\pi : X \to Y$ has P if

any affine open $U \subset Y$, $\pi^{-1}(U)$ has P

8.1 "Reasonable" classes of morphisms

A class of morphisms is called "reasonable" if:

(i) the class is preserved by composition

(ii) " base change

PRECISELY (need existence of fibre products) ↳ §10.1

$$\begin{tikzcd} X \times_Y Y' \arrow[r] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ Y' \arrow[r, "\forall"] & Y \end{tikzcd}$$

$$\boxed{\begin{array}{l} \pi \text{ in the class} \\ \Rightarrow \pi' \text{ in the class} \end{array}}$$

(iii) the class is local on the target

i.e. (a) $\pi : X \to Y$ in the class

$\Rightarrow$ $\forall$ open $V \subset Y$, $\pi|_{\pi^{-1}(V)} : \pi^{-1}(V) \to V$

is in the class

(b) $\cdot\ \pi : X \to Y$ a morphism

$\cdot\ \{V_i\}$ open cover of Y

s.t. $\pi\big|_{\pi^{-1}(V_i)} : \pi^{-1}(V_i) \to V_i$

is in the class $\forall i$

$\Rightarrow$ π is in the class

⚠ (i) + (ii) $\Rightarrow$ (iv) preserved by product EX 8.1.A.

i.e. $\left.\begin{array}{l} X \to Y \\ X' \to Y' \end{array}\right]$ S-schemes with prop. P

Then $X \times_S X' \longrightarrow Y \times_S Y'$ has prop. P

$\Rightarrow$ (v) Cancellation Theorem 11.2.1

"Easy" exercise 8.1.B. [class of isos of schemes] is "reasonable"

Def$^{\underline{n}}$ 8.1.2. $\pi : X \to Y$ is an **open embedding** (also open immersion)

if it is an open embedding of ringed spaces

i.e.

$$\begin{tikzcd}
\pi : (X, \mathcal{O}_X) \arrow[rr] \arrow[dr, "\tilde{\rho}"'] & & (Y, \mathcal{O}_Y) \\
& (U, \mathcal{O}_Y|_U) \arrow[ur, hook, "\tau"'] &
\end{tikzcd}$$

$\tau : U \hookrightarrow Y$ open

If $X \subset Y$ subset, and $\pi : X \to Y$ open embedding

then we call $(X, \mathcal{O}_X)$

an open subscheme of $(Y, \mathcal{O}_Y)$

$\mathbb{A}^2 \setminus \{0,0\} \to \mathbb{A}^2$

Q: $X \subset Y$ open, Y scheme $\Rightarrow$ X scheme?

YES: CLAIM: open affine subschemes are a base of the topology on Y

§8.2 Algebraic interlude

RECALL • $\phi: B \to A$ ring morphism

$a \in A$ is integral over B if:

$$a^n + ? a^{n-1} + \cdots + ? = 0$$

where coefficients lie in $\phi(B)$.

• ϕ is integral if every element of A is integral over $\phi(B)$

• if $\phi: B \to A$ integral + an inclusion of rings then ϕ is an integral extension

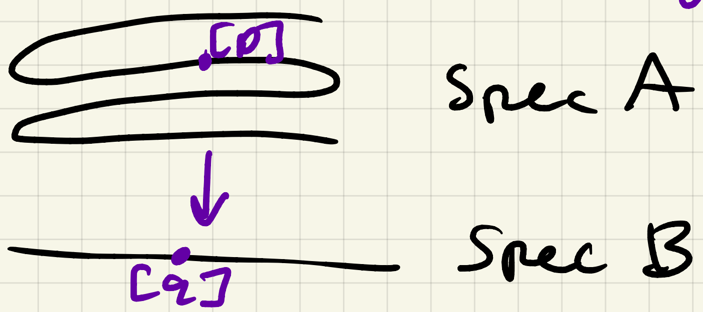
Thm 8.2.5 (The lying over theorem)

· $\phi: B \to A$ integral extension

Then $\forall$ prime ideal $q \subset B$ $\exists$ a prime ideal $p \in A$ s.t. $p \cap B = q$

GEOMETRIC TRANSLATION:

$=$ $\mathrm{Spec}\, A \longrightarrow \mathrm{Spec}\, B$ is surjective



Thm 8.2.F (Exercise) (Going-Up Theorem)

- $\phi: B \to A$ integral ring morphism

(1) $q_1 \subset q_2 \subset \cdots \subset q_n$ chain of primes in B

(2) $p_1 \subset p_2 \subset \cdots \subset p_m$ chain of primes in A $(1 \leq m < n)$ s.t. p_i "lies over" q_i

Then (2) extends to $p_1 \subset p_2 \subset \cdots \subset p_n$ s.t. p_i "lies over" q_i

(⚠ so Krull dim $B \leq$ Krull dim A)

Ex 8.2.H. (Nakayama V4)

- $(A, \underline{m})$ local ring
- M is fin. gen. A-module
- $f_1, \ldots, f_m \in M$ generate $M/\underline{m}M$

Then $f_1, \ldots, f_m$ generate M.

8.3 Finiteness conditions on morphisms

① Quasicompact

$\pi: X \to Y$ is quasi compact if $\forall$ open affine subset $U \subset Y$, $\pi^{-1}(U)$ is quasi compact

" every open cover has a finite subcover

⑦⑦ of affines

② Quasi separated

$\pi: X \to Y$ is quasi separated if $\forall$ open affine subset $U \subset Y$, $\pi^{-1}(U)$ is quasisep.

$\bigcap$ quasicompacts
fin.
$\|$
quasicompact

qcqs $\Rightarrow$ $\exists$ cover by fin. many affines
& intersection is covered by fin. many affines
etc.

① X scheme is quasicompact
$\iff$ $X \xrightarrow{\pi} \operatorname{Spec} \mathbb{Z}$ is quasicompact
→ think about it

③ Affine

$\pi : X \to Y$ is affine if
$\forall$ open affine $U \subset Y$, $\pi^{-1}(U)$ is an affine scheme,
⤷ as an open subscheme of X

Prop[n] 8.3.4 "affine" ness is affine local on the target

④ Finite

$\phi : B \to A$ ring morphism s.t.
A is a fin. gen. B-module
then say A is a finite B-algebra

$\pi : X \to Y$ is finite if
⚠ stronger than "fin-gen. B-algebra"

$\forall$ affine open $U = \operatorname{Spec} B \subset Y$,

$\pi^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$
where A : finite B-algebra.

SLOGAN: finite = closed + finite fibers

FACTS: finite morphisms are
- affine
- projective
- have finite fibers.

5 Integral

$\pi : X \to Y$ is integral if

- π is affine
- $\forall$ affine open $\operatorname{Spec} B \subset Y$,
  - $\pi^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$
  - induced map $B \to A$ is integral

(!) finite morphisms are integral
(!) integral morphisms are closed

6 Locally of finite type

$\pi : X \to Y$ is locally of fin. type if

- $\forall$ affine open $\operatorname{Spec} B \subset Y$
  $\forall$ affine open $\operatorname{Spec} A \subset \pi^{-1}(\operatorname{Spec} B)$

  ~~the induced~~ induced morphism $B \to A$ expresses A as a fin. gen. B-algebra

7 Finite type

$\pi : X \to Y$ is of finite type if

- π is locally of fin. type
- π is quasi compact.

<u>FACTS</u> · Finite = integral + finite type

· open embeddings are locally of finite type

EXAMPLE OF A FINITE MORPHISM

EX $$\operatorname{Spec} K[t] \longrightarrow \operatorname{Spec} K[u] \quad (K \text{ field!})$$

given by $\quad p(t) \longmapsfrom u$

$p(t)$ a degree n poly.

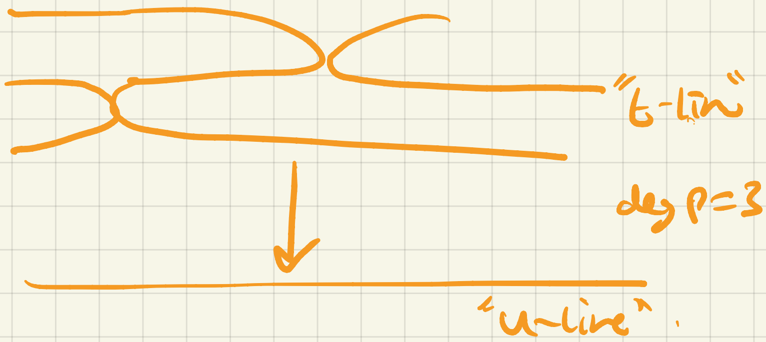
This finite:

$K[t]$ gen. as a $K[u]$-module

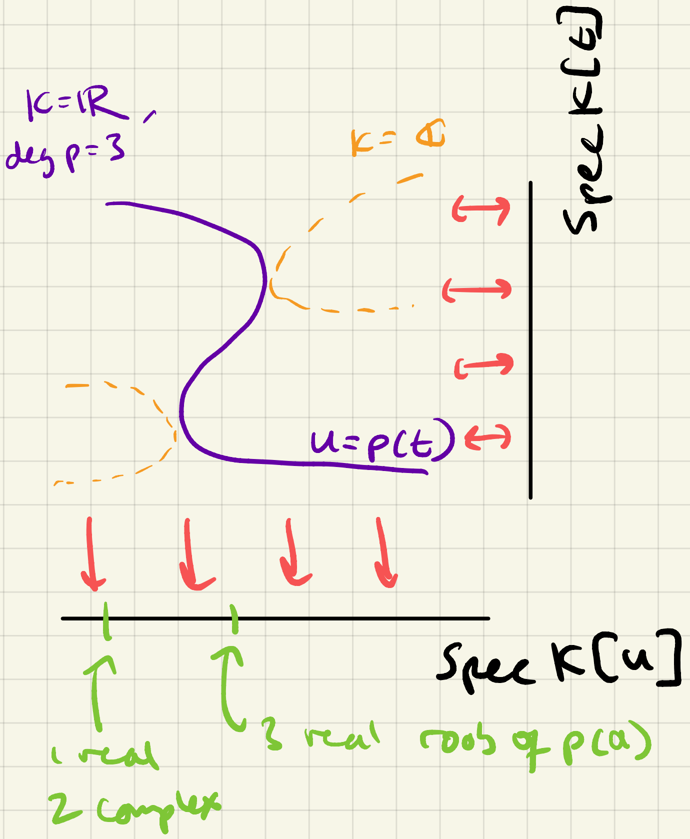
by $1, t, t^2, \dots, t^{n-1}$

$1 \in K[t]$

$u \cdot 1 := p(t)$



Unpacking the picture:



$$\begin{tikzcd} \dfrac{K[u,t]}{\langle u-p(t)\rangle} \arrow[rr, leftrightarrow, "\sim"] & & K[t] \\ & K[u] \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$\bar{t} \longmapsto t$

$\bar{u} \longmapsto p(t)$

$u \longmapsto \bar{u}$

$u \longmapsto p(t)$

$$K[t] \longleftarrow K[u]$$
$$p(t) \longleftarrow\!\shortmid\; u$$

$$\operatorname{Spec} K[t] \longrightarrow \operatorname{Spec} K[u]$$
$$(t-a) \longmapsto (u-p(a))$$

e.g. $p(t) = t^3$, $K = \mathbb{C}$

$$\left.\begin{array}{l}(t-a)\\(t-\zeta a)\\(t-\zeta^2 a)\end{array}\right] \longmapsto (u-a^3)$$

ζ: 3rd root of unity