

Anodynes and Fibrations

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$\# \in \{L, R, I\}$ $L = \text{left}, R = \text{right}, I = \text{inner}$

Def: A simplicial morphism $p: K \rightarrow S$ is called a $\#$ -fibration if it satisfies the RLP wrt to $\#$ -horn inclusions.

E.g.: Inner fibrations

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\quad} & K \\ \downarrow & \dots & \downarrow p \\ \Delta^n & \xrightarrow{\quad} & S \end{array} \quad \forall 0 < i < n$$

We write: $\# \text{Fib} = \{ \# \text{-fibrations} \} \subset \text{Mor}(s\text{Set})$
and $\text{Fib} := L\text{Fib} \cap R\text{Fib}$ is the set of (Kan) fibrations.

Conventions For $S \subset \text{Mor}(C)$ a set of morphisms in C .

$\chi_R(S) = \{ F \in \text{Mor}(C) \mid F \text{ has RLP wrt } S \}$

$\chi_L(S) = \{ F \in \text{Mor}(C) \mid F \text{ has LLP wrt } S \}$

"right orthogonal"
"left orthogonal"

and $\chi(S) := \chi_L(\chi_R(S))$

Example:

$\bullet C = s\text{Set}; S = \{ \# \text{-horn inclusions} \} \Rightarrow \chi_R(S) = \# \text{Fib}.$

Def: A $\#$ -anodyne map is a map which has the LLP wrt all $\#$ -fibrations, i.e.

$\# \in \{L, R, I, \emptyset\}$

$\# \text{-Anod} := \chi_L(\# \text{Fib}) = \chi(\# \text{-Horn Ind.})$

C : cocomplete (e.g. $s\text{Set}$)

Def: A set $S \subset \text{Mor}(C)$ is called **saturated** if it is closed under:

1) Isomorphisms: If $\Phi \in \text{Mor}(C)$ is iso, then $\Phi \in S$.

2) Pushouts:

For any pushout diagram:

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow i & & \downarrow i' \\ B & \longrightarrow & B' \end{array}$$

If $i \in S$, then $i' \in S$.

$$\begin{array}{ccc} A & \longrightarrow & A' \\ i \downarrow & & \downarrow i' \\ B & \longrightarrow & B' \end{array}$$

If $i \in S$, then $i' \in S$.

3) Retracts:

$$\begin{array}{ccccc} & & \text{id} & & \\ & \curvearrowright & & \curvearrowleft & \\ A' & \longrightarrow & A & \longrightarrow & A' \\ \downarrow i' & \parallel & \downarrow i & \parallel & \downarrow i' \\ B' & \longrightarrow & B & \longrightarrow & B' \\ & \curvearrowleft & & \curvearrowright & \\ & & \text{id} & & \end{array}$$

If $i \in S$, then $i' \in S$.

4) Countable colimits:

$$A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} \dots \quad (\mathbb{N}\text{-diagram in } \mathcal{C})$$

If $i_j \in S \forall j \in \mathbb{N}$, then the canonical map $i: A_0 \rightarrow \text{colim}_{\mathbb{N}} A_j$ is also in S .

(small)
5) Coproducts:

$$\{A_j \xrightarrow{i_j} B_j\}_{j \in J} \quad \text{small set of morphisms } i_j \in S.$$

Then, the canonical $\coprod i_j: \coprod A_j \rightarrow \coprod B_j$ is also in S .

Def: Let M be a set of morphisms in \mathcal{C} .

The **saturated closure** is defined as the smallest saturated set containing M and denoted by \overline{M} .

$$\overline{M} = \bigcap_{\substack{M \subset S \\ S \text{ saturated}}} S$$

Lemma (1.3.8) (\mathcal{C} : cocomplete)

Let $M \subset \text{Mor}(\mathcal{C})$ be a set of morphisms. Then, $\chi_{\mathcal{C}}(M)$ is saturated.

PF: 1) Iso $\varphi: X \rightarrow Y$

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & K \\ \varphi \downarrow & \dashrightarrow & \downarrow \in M \\ Y & \longrightarrow & S \end{array}$$

The lifting solution $\Rightarrow \varphi \in \chi_{\mathcal{C}}(M)$.
is provided by $\alpha \circ \varphi^{-1}$ ✓

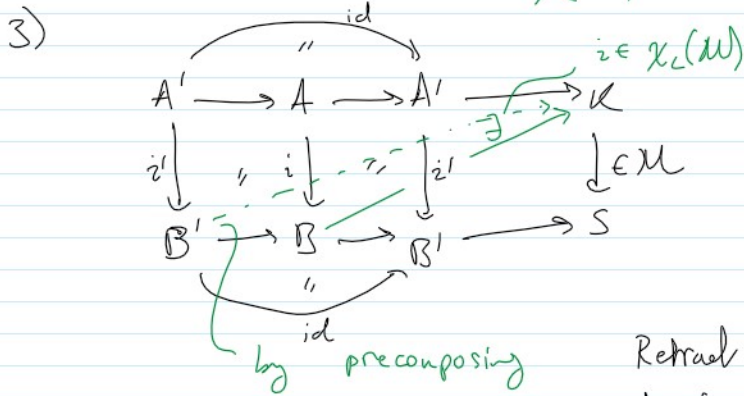
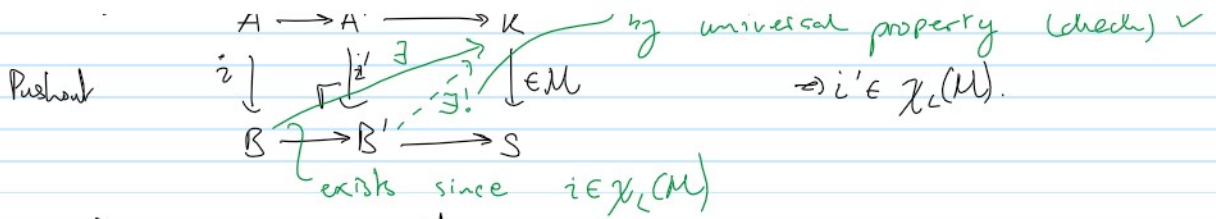
2)

$$\begin{array}{ccccc} A & \longrightarrow & A' & \longrightarrow & K \\ i \downarrow & \dashrightarrow & \downarrow i' & \dashrightarrow & \downarrow \in M \end{array}$$

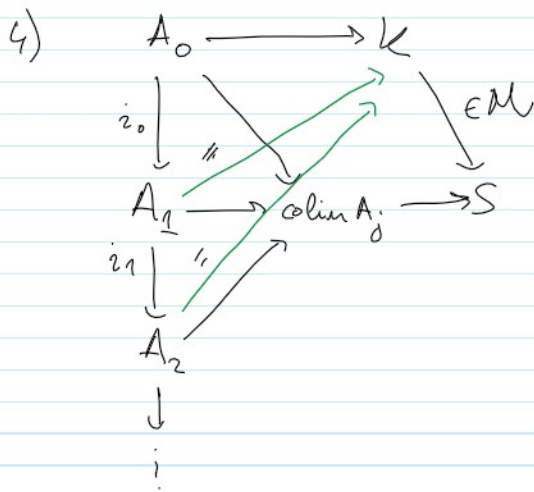
by universal property (check) ✓

Pushout

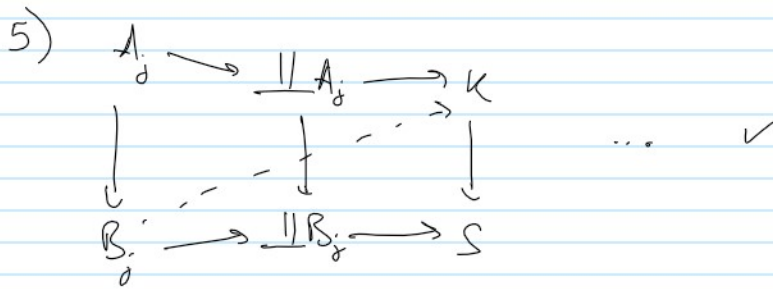
$\Rightarrow i' \in \chi_{\mathcal{C}}(M)$.



Retraal property provides solution to the right square! \checkmark



The solutions provided by i_j give a cone under v. vertex K .
 $\Rightarrow \exists! \text{ colim } A_j \rightarrow K$ (check) everything works \checkmark



□

Cor: $\bar{\mathcal{U}} \subset \mathcal{X}(\mathcal{M})$

Pf: By def: $\mathcal{U} \subset \mathcal{X}(\mathcal{M})$ and both are saturated \square .

Def: An obj K in \mathcal{C} is called **compact** if for every filtered diagram:

$$I \rightarrow \mathcal{C}$$

$$i \mapsto X_i$$

and every morphism $f: K \rightarrow \text{colim } X_i$ it holds:

(i) $\exists i \in I$ and $f_i: K \rightarrow X_i$ s.t.

$$\begin{array}{ccc}
 K & \xrightarrow{f_i} & X_i \\
 & \searrow f & \downarrow \\
 & & \text{colim } X_j
 \end{array}$$

(ii) Given $i, j \in I$ and f_i, f_j as above

Then $\exists k \geq i, j$ s.t.

$$\begin{array}{ccc}
 K & \xrightarrow{f_i} & X_i \\
 f_i \downarrow & \cong & \downarrow \\
 X_j & \longrightarrow & X_k
 \end{array}$$

Rem: If \mathcal{C} is locally small, an object K is compact iff $\text{Hom}_{\mathcal{C}}(K, -): \mathcal{C} \rightarrow \text{Set}$ commutes w. filtered colimits.

Examples:

- A set K in Set is compact iff it is finite.
- A subset K is compact iff it has finitely many non-degenerate simplices.

Prop (1.3.9) (Small Object Argument)

\mathcal{C} : locally small, ~~complete~~ cocomplete. Suppose $\mathcal{M} \subset \text{Mor}(\mathcal{C})$ where all sources are compact objects in \mathcal{C} . Then, any morphism $f: X \rightarrow Y$ in \mathcal{C} admits a factorization

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 h \downarrow & \cong & \uparrow g \\
 & Z &
 \end{array}$$

by $g \in \mathcal{X}_{\mathcal{R}}(\mathcal{M})$ and $h \in \mathcal{M}$.

Pf:

Consider the small set

$$\left\{ \begin{array}{ccc} A_j & \longrightarrow & X \\ i_j \downarrow & \cong & \downarrow f \\ B_j & \longrightarrow & Y \end{array} \right\}_{j \in J}$$

of lifting problems of f w. $i_j \in \mathcal{M}$.

$$\mathcal{M} = \chi(\mathcal{M})$$

PF: $\bar{\mathcal{M}} \subset \chi(\mathcal{M})$ was true in general.

• let $f \in \chi(\mathcal{M})$. \exists factorization $f = g \circ h$ w. $g \in \chi_{\mathcal{R}}(\mathcal{M})$, $h \in \bar{\mathcal{M}}$

$$\begin{array}{ccc} X & \xrightarrow{\quad} & Z \\ f \downarrow & \nearrow h & \downarrow g \\ Y & \xrightarrow{\quad} & Y \end{array} \quad f \in \chi(\mathcal{M})$$

w) Consider

$$\begin{array}{ccccc} X & \xlongequal{\quad} & X & \xlongequal{\quad} & X \\ f \downarrow & & \downarrow h & \text{,} & \downarrow g \\ Y & \xrightarrow{\quad} & Z & \xrightarrow{\quad} & Y \\ & & \text{id} & & \end{array}$$

$$h \in \bar{\mathcal{M}} \Rightarrow f \in \bar{\mathcal{M}}.$$



Therefore:

$$\# \text{-Anod} \stackrel{!}{=} \overline{(\# \text{-Horns})}$$

and therefore $\# \text{-Anod} \subset \text{Monos}$.

↑
Monos are saturated!

Prop (4.3.12)

$\overline{(\{I^n \hookrightarrow \Delta^n\})} \neq \text{IAnod}$ and not all composes are ex-acts.

PF: (Sketch)

$$I_1^3 \hookrightarrow \Delta^3 \quad \text{By Small Obj. Argument } \exists \begin{array}{ccc} I_1^3 & \hookrightarrow & \Delta^3 \\ & \searrow & \nearrow \\ & X & \end{array}$$

$$\text{w. } I_1^3 \rightarrow X \in \overline{\{\text{spine ind.}\}}$$

$$\text{and } X \rightarrow \Delta^3 \in \chi_{\mathcal{R}}(\text{spine ind.}).$$

$$= \chi_{\mathcal{R}}(\overline{\{\text{spine ind.}\}})$$

Claim X is a composer:

$$\begin{array}{ccc} I^n & \longrightarrow & X \\ \downarrow & & \\ I^n & & \end{array}$$

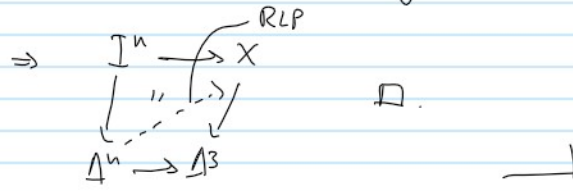
Consider

$$\begin{array}{ccc} I^n & \longrightarrow & X \\ \downarrow & & \downarrow \\ I^n \dots \rightarrow & & \Delta^3 \end{array}$$

since Δ^3 is a composer.

Δ^n

$\Delta^n \dots \rightarrow \Delta^3$ since Δ^3 is a cospan.

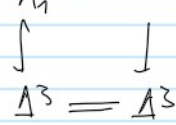


Suppose now: $\overline{\{\text{Spine Incl}\}} = \text{IAncd}$.

Then X has RLP wrt all inner horns.



but $A_1^3 \rightarrow X$ has no solution (induction) \square



Remark: $\overline{\{\text{Spine Incl}\}} \subsetneq \text{IAncd}$

Cor (1.3.18)

$F: C \rightarrow D$ w. C, D simplicial cat. s.t. on Thom's $\text{Hom}_C(X, Y) \rightarrow \text{Hom}_D(FX, FY)$ is Kan fibration. Then, $NF: NC \rightarrow ND$ is an inner fibration.

Rem: • This extends the lemma from previously that Kan-enriched cats give ∞ -cats (nerve).

• In particular, for ordinary functors.

(In the model structures, anodynes will correspond to weak equivalences in Top)

Prop (1.3.22) (w/o p.f)

Spine inclusions are inner anodyne.

Cor: ∞ -cats are cospanners.



$I^n \rightarrow C$

\dots

□

$$\begin{array}{ccc} \mathbb{I}^n & \longrightarrow & \mathcal{C} \\ \downarrow & \dashrightarrow & \downarrow \\ \Delta^n & \longrightarrow & \mathcal{D} \end{array}$$

using $\mathbb{I}Fib \stackrel{def}{=} \mathcal{X}_R(\mathbb{I}\text{-Horns}) = \mathcal{X}_R(\mathbb{I}Anod)$ □

Def A **trivial fibration** is a map in $\mathcal{X}_R(\{\partial\Delta^n \hookrightarrow \Delta^n\})$

Def: • Let \mathcal{J} be the category: $ob(\mathcal{J}) = \{a, b\}$
 $Hom_{\mathcal{J}}(a, b) = Hom_{\mathcal{J}}(b, a) = * = End_{\mathcal{J}}(a) = End_{\mathcal{J}}(b)$

• An inner fibration $f: \mathcal{C} \rightarrow \mathcal{D}$ between $\mathcal{C}, \mathcal{D} \in \mathfrak{w}\text{-Cat}$ is called **local fibration** if it has RLP art $\Delta^0 \rightarrow \mathcal{J}$.

$$\begin{array}{ccc} \Delta^0 & \longrightarrow & \mathcal{C} \\ \downarrow & \dashrightarrow & \downarrow f \\ \mathcal{J} & \longrightarrow & \mathcal{D} \end{array}$$

$$\begin{array}{ccc} id & \xrightarrow{f} & id \\ \downarrow & \circlearrowleft & \downarrow \\ a & \xrightarrow{f} & b \end{array}$$

Smash Products

smash (or \boxtimes)

$$f: A \rightarrow A'; g: B \rightarrow B' \quad f \boxtimes g: A \times B' \underset{A \times B}{\parallel} A' \times B \longrightarrow A' \times B'$$

induced by the pushout:

$$\begin{array}{ccc} A \times B & \xrightarrow{f \times id} & A' \times B \\ id \times g \downarrow & \lrcorner & \downarrow id \times g \\ A \times B' & \xrightarrow{f \times id} & A' \times B' \end{array}$$

Examples: $f: * \rightarrow X, g: * \rightarrow Y$.

Prop:

- 1) If f, g are monic, then $f \boxtimes g$ is monic
- 2) \boxtimes is associative $f \boxtimes (g \boxtimes h) \simeq (f \boxtimes g) \boxtimes h$.

Lem (1.3.34)

$$\begin{aligned}
 \Gamma \text{Anal} &:= \chi(\text{Inner horns}^{\text{incl.}}) = \chi(\{(\mathcal{K} \hookrightarrow \mathcal{S}) \boxtimes (\Lambda_1^2 \hookrightarrow \Delta^2) \mid \mathcal{K} \hookrightarrow \mathcal{S}\}) \\
 &= \chi(\{(\partial \Delta^n \hookrightarrow \Delta^n) \boxtimes (\Lambda_1^2 \hookrightarrow \Delta^2)\}) \\
 &= \chi(\{(\mathcal{K} \hookrightarrow \mathcal{S}) \boxtimes \{\Lambda_{\text{occ}}^n \hookrightarrow \Delta\}\})
 \end{aligned}$$

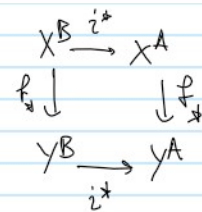
Rem: Same applies for $\mathcal{L}\text{Anal}$ and $\mathcal{R}\text{Anal}$

$\left. \begin{array}{l} \text{replace: } \rightsquigarrow \text{ left horns} \\ \Lambda_1^2 \hookrightarrow \Delta^2 \rightsquigarrow \{0\} \rightarrow \Delta^1 \end{array} \right\} \rightsquigarrow \text{right horns}$
 $\Lambda_1^2 \hookrightarrow \Delta^2 \rightsquigarrow \{1\} \rightarrow \Delta^2$

Lem: let g be mono. Then
 If f is $\#$ -analytic, so is $f \boxtimes g$

Now for $f: X \rightarrow Y$; $i: A \rightarrow B$

$\langle f, i \rangle : X^B \rightarrow X^A \times_{Y^A} Y^B$ by



Theorem

$f \in \#$ -Fib, i monic. Then.

- 1) $\langle f, i \rangle$ is $\#$ -Fib.
- 2) If $i \in \#$ -Anal, then f trivial Fibration

Cor: If X is ω -cat, then X^k is also an ω -cat.
 (also true for Ken complexes)

Funder: Cat: $\mathcal{C}, \mathcal{D} \in \omega\text{-Cat}$. $\text{Hom}(\mathcal{C}, \mathcal{D})$ is the ω -cat of functors.

$\mathcal{C}: \omega\text{-cat}$.

let $x, y \in \mathcal{C}$.

$$\begin{array}{ccc} \text{map}_{\mathcal{C}}(x, y) & \longrightarrow & \text{Hom}(\Delta^1, \mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & \mathcal{C} \times \mathcal{C} \end{array}$$

Prop: is ω -groupoid.