Anodynes and Fibrations

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$\# \in \{L, R, I\}$ L = left, R = right, I = inner

Def: A simplicial morphism $p: K \to S$ is called a #-fibration if it satisfies the RLP wrt to #-horn inclusions.

E.g.: Inner fibration

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & K \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd} \quad \forall\, 0<i<n$$

We write: $\#Fib = \{\#\text{-fibrations}\} \subset Mor(sSet)$

and $Fib := LFib \cap RFib$ is the set of (Kan) fibrations.

Convention: For $S \subset Mor(\mathcal{C})$ a set of morphisms in $\mathcal{C}$.

$\chi_R(S) = \{f \in Mor(\mathcal{C}) \mid f \text{ has RLP wrt } S\}$ "right orthogonal"
$\chi_L(S) = \{f \in Mor(\mathcal{C}) \mid f \text{ has LLP wrt } S\}$ "left orthogonal"

and $\boxed{\chi(S) := \chi_L(\chi_R(S))}$

Example:

- $\mathcal{C} = sSet$; $S = \{\#\text{-horn inclusions}\} \Rightarrow \chi_R(S) = \#Fib$.

Def: A #-anodyne map is a map which has the LLP wrt all #-fibrations, i.e. $\# \in \{L, R, I, \emptyset\}$

$$\#\text{-}Anod := \chi_L(\#Fib) = \chi(\#\text{-Horn Ind.})$$

$\mathcal{C}$: cocomplete (e.g. sSet)

Def: A set $S \subset Mor(\mathcal{C})$ is called saturated if it is closed under:

1) Isomorphisms: If $\Phi \in Mor(\mathcal{C})$ is iso, then $\Phi \in S$.
2) Pushouts:

For any pushout diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\ B \arrow[r] & B' \end{tikzcd}$$

If $i \in S$, then $i' \in S$.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\ B \arrow[r] & B' \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

If $i \in S$, then $i' \in S$.

3) Retracts:

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "i'"] \arrow[rr, bend left, "id"] & A \arrow[r] \arrow[d, "i"] & A' \arrow[d, "i'"] \\ B' \arrow[r] \arrow[rr, bend right, "id"'] & B \arrow[r] & B' \end{tikzcd}$$

If $i \in S$, then $i' \in S$.

4) Countable colimits:

$A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} \cdots$ ($\mathbb{N}$-diagram in $\mathcal{C}$)

If $i_j \in S$ $\forall j \in \mathbb{N}$, then the canonical map $i : A_0 \longrightarrow \operatorname{colim}_{\mathbb{N}} A_j$ is also in S.

5) (small) Coproducts:

$\{A_j \xrightarrow{i_j} B_j\}_{j \in J}$ small set of morphis $i_j \in S$.

Then, the canonical $\coprod i_j : \coprod A_j \longrightarrow \coprod B_j$ is also in S.

Def: Let $\mathcal{M}$ be a set of morphisms in $\mathcal{C}$.
The saturated closure is defined as the smallest saturated set containing $\mathcal{M}$ and denoted by $\overline{\mathcal{M}}$.

$$\rightsquigarrow \boxed{\overline{\mathcal{M}} = \bigcap_{\substack{\mathcal{M} \subset S \\ \text{saturated}}} S}$$

Lem: (1.3.8) ($\mathcal{C}$: cocomplete)
Let $\mathcal{M} \subset Mor(\mathcal{C})$ be a set of morphisms.
Then, $\chi_L(\mathcal{M})$ is saturated.

Pf: 1) Iso $\varphi : X \to Y$

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & K \arrow[d, "\in \mathcal{M}"] \\ Y \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

The lifting solution is provided by $\alpha \circ \varphi^{-1}$ ✓ $\Rightarrow \varphi \in \chi_L(\mathcal{M})$.

2)

Pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & A' \arrow[r] \arrow[d, "i'"] & K \arrow[d, "\in \mathcal{M}"] \\ {} & {} \arrow[ur, "\exists"] \arrow[ur, dashed, "\exists !"'] & {} \end{tikzcd}$$

by universal property (check) ✓

$\Rightarrow i' \in \chi_L(\mathcal{M})$.

Pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & A' \arrow[r] \arrow[d, "i'"] & K \arrow[d, "\in \mathcal{M}"] \\ B \arrow[r] \arrow[urr, "\exists"] & B' \arrow[r] \arrow[ur, dashed, "\exists!"] \arrow[ul, phantom, "\ulcorner"] & S \end{tikzcd}$$

by universal property (check) ✓

$\Rightarrow i' \in \chi_L(\mathcal{M})$.

exists since $i \in \chi_L(\mathcal{M})$

3)

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "i'"'] \arrow[rr, bend left, "id"] & A \arrow[r] \arrow[d, "i"] & A' \arrow[r] \arrow[d, "i'"] & K \arrow[d, "\in \mathcal{M}"] \\ B' \arrow[r] \arrow[rr, bend right, "id"'] & B \arrow[r] \arrow[urr, "\exists"] & B' \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

$i \in \chi_L(\mathcal{M})$

by preconposing

Retract property provides solution to the right square! ✓

4)

$$\begin{tikzcd} A_0 \arrow[rr] \arrow[d, "i_0"'] \arrow[dr] & & K \arrow[d, "\in \mathcal{M}"] \\ A_1 \arrow[r] \arrow[d, "i_1"'] \arrow[urr] & \operatorname{colim} A_j \arrow[r] \arrow[ur] & S \\ A_2 \arrow[ur] \arrow[uurr] \arrow[d] & & \\ \vdots & & \end{tikzcd}$$

The solutions provided by i_j give a cone under w. vertex K.

$\Rightarrow \exists! \operatorname{colim} A_j \to K$. (check) everything works ✓

5)

$$\begin{tikzcd} A_j \arrow[r] \arrow[d] & \coprod A_j \arrow[r] \arrow[d] & K \arrow[d] \\ B_j \arrow[r] \arrow[urr, dashed] & \coprod B_j \arrow[r] & S \end{tikzcd}$$

... ✓

□.

Cor: $\overline{\mathcal{M}} \subset \chi(\mathcal{M})$

Pf: By def: $\mathcal{M} \subset \chi(\mathcal{M})$ and both are saturated □.

Def: An obj K in $\mathcal{C}$ is called compact if for every filtered diagram:
$$\begin{aligned} I &\to \mathcal{C} \\ i &\mapsto X_i \end{aligned}$$
and every morphism: $f: K \to \operatorname{colim} X_i$ it holds:

(i) $\exists i \in I$ and $f_i : K \to X_i$ s.t.
$$\begin{tikzcd} K \arrow[r, "f_i"] \arrow[dr, "f"'] & X_i \arrow[d] \arrow[dl, phantom, "="] \\ & \operatorname{colim} X_j \end{tikzcd}$$

(ii) Given $i, j \in I$ and f_i, f_j as above

Then $\exists k \geq i, j$ s.t.
$$\begin{tikzcd} K \arrow[r, "f_i"] \arrow[d, "f_j"'] \arrow[dr, phantom, "="] & X_i \arrow[d] \\ X_j \arrow[r] & X_k \end{tikzcd}$$

Rem: If $\mathcal{C}$ is locally small, a object K is compact iff $\mathrm{Hom}_{\mathcal{C}}(K, -) : \mathcal{C} \to \mathrm{Set}$ commutes w. filtered colimits.

Example:
- A set K in Set is compact iff it is finite.
- A sSet K is compact iff it has finitely many non-degenerate simplices.

Prop (1.3.9) (Small Object Argument)
$\mathcal{C}$: locally small, cocomplete. Suppose $\mathcal{M} \subset \mathrm{Mor}(\mathcal{C})$ where all sources are compact objects in $\mathcal{C}$. Then, any morphism $f: X \to Y$ in $\mathcal{C}$ admits a factorization
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & \arrow[d, phantom, "="] & Y \\ & Z \arrow[ur, "g"'] & \end{tikzcd}$$
by $g \in \chi_R(\mathcal{M})$ and $h \in \overline{\mathcal{M}}$.

Pf: Consider the small set
$$\left\{ \begin{tikzcd} A_j \arrow[r] \arrow[d, "i_j"'] \arrow[dr, phantom, "="] & X \arrow[d, "f"] \\ B_j \arrow[r] & Y \end{tikzcd} \right\}_{j \in J}$$
of left lifting problems of f w. $i_j \in \mathcal{M}$.

$$\left\{ \begin{tikzcd} A_j \arrow[r] \arrow[d, "i_j"'] \arrow[dr, phantom, "="] & X \arrow[d, "f"] \\ B_j \arrow[r] & Y \end{tikzcd} \right\}_{j \in J}$$ of lifting problems of f w. $i_j \in \mathcal{M}$.

Then: taking the pushout

$$\begin{tikzcd} \coprod A_j \arrow[r] \arrow[d, "\coprod i_j"'] \arrow[dr, phantom, "="] & X \arrow[d, "h_1"'] \arrow[dd, bend left, "f"] \\ \coprod B_j \arrow[r] \arrow[dr, bend right, "="'] & X_1 \arrow[d, "g_1"', "\exists!"] \\ & Y \end{tikzcd}$$

w. $h_1 \in \overline{\mathcal{M}}$ (closed under coprods & pushouts)

Repeat for g_1:

$$\begin{tikzcd} \coprod A_j \arrow[r] \arrow[d] & X_1 \arrow[d, "h_2"'] \arrow[dd, bend left, "g_1"] \\ \coprod B_j \arrow[r] \arrow[dr] & X_2 \arrow[d, "g_2"'] \\ & Y \end{tikzcd}$$

$h_2 \in \overline{\mathcal{M}}$ and $f = g_1 \circ h_1 = g_2 \circ h_2 \circ h_1$

Inductively:

$$\begin{tikzcd} X \arrow[r, "h_1"] \arrow[dr, "f"'] & X_1 \arrow[r, "h_2"] \arrow[d, "g_1"] & X_2 \arrow[r, "h_3"] \arrow[dl, "g_2"] & \cdots \\ & Y & & \end{tikzcd}$$

$h_j \in \overline{\mathcal{M}}$ and Y a cone under

$\Rightarrow \exists!\ \operatorname{colim} X_j \xrightarrow{g} Y$

s.t.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & {} \arrow[d, phantom, "="] & Y \\ & \operatorname{colim} X_i \arrow[ur, "g"'] & \end{tikzcd}$$

$\overline{\mathcal{M}} \ni h$

saturated

(now we need compactness)

<u>Claim:</u> $g \in \chi_R(\mathcal{M})$.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathcal{M} \ni i"'] & \operatorname{colim} X_i \arrow[d, "g"] \\ B \arrow[r] & Y \end{tikzcd}$$

By assumption, A is compact

compactness

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] \arrow[dd, "i"'] & & \operatorname{colim} X_i \arrow[dd, "g"] \\ & X_k \arrow[ur] \arrow[d] & \\ B \arrow[r] \arrow[rr, bend right] & X_{k+1} \arrow[uur] \arrow[r] & Y \end{tikzcd}$$

compatible by compactness.

$\square$

Cor: Under the same hypothesis:

$$\overline{\mathcal{M}} = \chi(\mathcal{M})$$

Pf: • $\overline{\mathcal{M}} \subset \chi(\mathcal{M})$ was true in general.

$\mathcal{M} = \chi(\mathcal{M})$

Pf: • $\overline{\mathcal{M}} \subset \chi(\mathcal{M})$ was true in general.

• let $f \in \chi(\mathcal{M})$. $\exists$ factorization $f = g \circ h$ w. $g \in \chi_R(\mathcal{M})$, $h \in \overline{\mathcal{M}}$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, equal] \arrow[ur, "\exists"] & Y \end{tikzcd} \qquad f \in \chi(\mathcal{M})$$

$\rightsquigarrow$ Consider

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r, equal] \arrow[d, "h"] & X \arrow[d, "f"] \\ Y \arrow[r] \arrow[rr, bend right, "id"'] & Z \arrow[r, "g"] & Y \end{tikzcd} \qquad h \in \overline{\mathcal{M}} \Rightarrow f \in \overline{\mathcal{M}}.$$

$\square$

Therefore:

$$\#\text{-Anod} \overset{!}{=} \overline{(\#\text{-Horns})}$$

and therefore $\#\text{-Anod} \subset \text{Monos}$.

Monos are saturated!

Prop (1.3.12)

$\overline{(\{I^n \hookrightarrow \Delta^n\})} \neq \text{IAnod}$ and not all composers are ∞-cats.

Pf: (Sketch)

$\Lambda^3_1 \hookrightarrow \Delta^3$ By Small Obj. Argument $\exists$

$$\begin{tikzcd} \Lambda^3_1 \arrow[rr] \arrow[dr] & & \Delta^3 \\ & X \arrow[ur] & \end{tikzcd}$$

w. $\Lambda^3_1 \to X \in \overline{\{\text{spine incl.}\}}$

and $X \to \Delta^3 \in \chi_R(\text{spine incl.})$

$= \chi_R(\overline{\{\text{spine incl}\}})$

Claim X is a composer:

$$\begin{tikzcd} I^n \arrow[r] \arrow[d, hook] & X \\ \Delta^n & \end{tikzcd}$$

Consider

$$\begin{tikzcd} I^n \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^n \arrow[r, dashed, "\exists"'] & \Delta^3 \end{tikzcd}$$

since Δ^3 is a composer.

Λ^n $\qquad\qquad$ $\Delta^n \dashrightarrow \Delta^3$ since Δ^3 is a composer.

$$\Rightarrow \quad \begin{tikzcd} I^n \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^3 \end{tikzcd} \quad \text{RLP} \qquad \square.$$

Suppose now: $\overline{\{\text{spine ind.}\}} = \text{IAnod}$.

Then $\begin{array}{c} X \\ \downarrow \\ \Delta^3 \end{array}$ has RLP wrt all inner horns.

but
$$\begin{tikzcd} \Lambda^3_1 \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^3 \arrow[r, equal] & \Delta^3 \end{tikzcd}$$
has no solution (induction) Skip

$\square$

Remark: $\overline{\{\text{Spine Ind}\}} \subsetneq \text{IAnod}$

Cor (1.3.18)

$F : \mathcal{C} \to \mathcal{D}$ w. $\mathcal{C}, \mathcal{D}$ simplicial cat. s.t. on Hom's $\text{Hom}_{\mathcal{C}}(X,Y) \to \text{Hom}_{\mathcal{D}}(FX, FY)$ is Kan fibration.

Then, $NF : N\mathcal{C} \to N\mathcal{D}$ is an inner fibration.

Rem: • This extends the lemma from previously that Kan-enriched cats give ∞-cats (nerve).

• In particular, for ordinary functors.

(In the model structures, anodynes will correspond to weak equivalences in Top)

Prop (1.3.22) (w/o pf)

Spine inclusions are <u>inner anodyne</u>.

Cor: ∞-cats are composers.

$I^n \longrightarrow \mathcal{C}$ $\qquad\qquad$ I^{II} def $\ldots$

$$\begin{tikzcd} I^n \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

using $\mathrm{IFib} \overset{\text{def}}{=} \chi_R(\text{I-Horns}) = \chi_R(\text{I Anod})$

$\square$

Def A trivial fibration is a map in $\chi_R(\{\partial\Delta^n \hookrightarrow \Delta^n\})$

Def: ∘ Let $\mathcal{J}$ be the category: $ob(\mathcal{J}) = \{a, b\}$

$\mathrm{Hom}_{\mathcal{J}}(a,b) = \mathrm{Hom}_{\mathcal{J}}(b,a) = * = \mathrm{End}_{\mathcal{J}}(a) = \mathrm{End}_{\mathcal{J}}(b)$

$$\begin{tikzcd} a \arrow[loop left, "id"] \arrow[r, bend left, "\varphi"] & b \arrow[l, bend left, "\varphi^{-1}"] \arrow[loop right, "id"] \end{tikzcd}$$

∘ An inner fibration $f: \mathcal{C} \to \mathcal{D}$ between $\mathcal{C}, \mathcal{D} \in \infty\text{-Cats.}$ is called Joyal fibration if it has RLP wrt $\Delta^0 \to \mathcal{J}$.

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "f"] \\ \mathcal{J} \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

Smash Products

$f: A \to A'$; $g: B \to B'$ smash (or $f \wedge g$) $f \boxtimes g : A \times B' \coprod_{A \times B} A' \times B \longrightarrow A' \times B'$

induced by the pushout:

$$\begin{tikzcd} A \times B \arrow[r, "f \times id"] \arrow[d, "id \times g"'] \arrow[dr, phantom, "\lrcorner"] & A' \times B \arrow[d, "id \times g"] \\ A \times B' \arrow[r, "f \times id"'] & A' \times B' \end{tikzcd}$$

Example: $f: * \to X$, $g: * \to Y$.

Prop:
1) If f, g are monic, then $f \boxtimes g$ is monic
2) $\boxtimes$ is associative: $f \boxtimes (g \boxtimes h) \simeq (f \boxtimes g) \boxtimes h$.

Lem (1.3.34)

$$\mathrm{IAnod} := \chi(\text{Inner horns}^{\text{incl.}}) = \chi(\{(K \hookrightarrow S) \boxtimes (\Lambda^2_1 \hookrightarrow \Delta^2) \mid K \hookrightarrow S\})$$

$$= \chi(\{(\partial\Delta^n \hookrightarrow \Delta^n) \boxtimes (\Lambda^2_1 \hookrightarrow \Delta^2)\})$$

$$= \chi(\{(K \hookrightarrow S) \boxtimes \{\Lambda^n_{i, 0<i<n} \hookrightarrow \Delta\}\})$$

Rem: Same applies for LAnod and RAnod

replace: ⇝ left horns; $\Lambda^2_1 \hookrightarrow \Delta^2 \rightsquigarrow \{0\} \to \Delta^1$ | ⇝ right horns; $\Lambda^2_1 \hookrightarrow \Delta^2 \rightsquigarrow \{1\} \to \Delta^1$

Lem: Let g be mono. Then
If f is #-anodyne, so is $f \boxtimes g$

Now for $f: X \to Y$; $i: A \to B$

$$\langle f, i\rangle : X^B \longrightarrow X^A \times_{Y^A} Y^B \quad \text{by} \quad \begin{matrix} X^B & \xrightarrow{i^*} & X^A \\ f_* \downarrow & & \downarrow f_* \\ Y^B & \xrightarrow[i^*]{} & Y^A \end{matrix}$$

Theorem

$f \in$ #-Fib, i monic. Then.

1) $\langle f, i\rangle$ is #-Fib.

2) If $i \in$ #-Anod, then f trivial Fibration

Cor: If X is ∞-cat, then X^K is also an ∞-cat.
(also true for Kan complexes)

Functor Cat: $\mathcal{C}, \mathcal{D} \in \infty\text{-Cat}$. $\rightsquigarrow$ $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ is the ∞-cat of functors.

$\mathcal{C}$: ∞-cat.

let $x, y \in \mathcal{C}$.

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}(\Delta^1, \mathcal{C}) \arrow[d] \\
* \arrow[r] & \mathcal{C} \times \mathcal{C}
\end{tikzcd}$$

Prop: is ∞-groupoid.