Eisenboaties 24/3/23 - Toward Schemes

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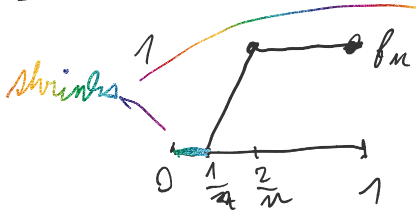
Recall last time: examples of sheaves $C(X)$, $C^\infty(X)$

Prop: If X is a mfd., $V \subset X$ open, $C(X)$ or $C^\infty(X)$ is never Noetherian

[R is Noetherian if any ideal $I \subseteq R$ is f.g $\iff$ if any ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ stabilises]

Pf $C([0,1])$ is not Noeth $(f_1) \subseteq (f_2) \subseteq (f_3) \subseteq \ldots$



$f\, C([0,1]) = \{ g \in C([0,1])$

So we want to look at "nicer" fns.: (real or) complex analytic fns.

Ex: On $\mathbb{CP}^1$, there are no non-constant an. fns. [Liouville's thm] $\Rightarrow$ we need to think about them open by open

$$C(X) \supseteq C^\infty(X) \supseteq C_{hol}(X) \underset{\text{(sometimes)}}{\supseteq} (\text{polynomial fns.})$$

Prop: If $f: \mathbb{C} \to \mathbb{C}$ is a hol. fn. bounded by a poly., then f is a polynomial

What information can one recover from knowing the fns. on X?

Ex: If X, Y are smooth mfds., $f: X \to Y$ is cts. $f^*: C(Y) \to C(X)$

f is smooth $\iff f^*(C^\infty(Y)) \subseteq C^\infty(X)$

(for $\Leftarrow$: consider local coord. fns. on Y)

If f is smooth, $f^*: \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$. moreover: $m_{Y,f(x)} \subseteq \mathcal{O}_{Y,f(x)}$

$\{ f$" vanishing at $f(x) \}$

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$m_{Y,f(x)} = \{\text{fns. vanishing at } f(x)\}$

$f^*(m_{Y,f(x)}) \subseteq m_{X,x} \implies$ get an induced map.

$$f^*: \underbrace{m_{Y,f(x)}/m^2_{Y,f(x)}}_{= T^*_{f(x)}Y} \longrightarrow \underbrace{m_{X,x}/m^2_{X,x}}_{= T^*_x X}$$

Defining Spec

Ex: $R = \mathbb{C}[x] \rightsquigarrow$ what is the space X s.t. R = ring of fns. on X (in same sense, "naturally")

Guess: $X = \mathbb{C}$ (not quite)

How to get information between X and R?

Note that we have bijection $\{a \in \mathbb{C}\} \longleftrightarrow \{\text{principal ideal } (x-a) \subseteq R\}$

In fact, all the prime ideals of R are (0), $(x-a)$, so we have:

$$\text{pts in } X \longleftrightarrow \text{prime ideals of } R$$

Given $x \in X$, we have a map $ev_x : R \longrightarrow \mathbb{C}$

$$f \longmapsto f(x)$$

$$0 \to (x-a) \longrightarrow \mathbb{C}[x] \xrightarrow{ev_a} \mathbb{C} \longrightarrow 0$$

We know how to evaluate f on a prime ideal $(x-a)$:

we know how to evaluate f at a prime ideal ...

map it $\underbrace{f \pmod{(x-a)}}$

Ex: What if $R = \mathbb{Z}$?

$X = \{\text{prime ideals of } \mathbb{Z}\} = \{(p) : p \text{ prime}\} \cup \{(0)\}$

Evaluating $n \in \mathbb{Z}$ at (p) is just reducing it modulo p

$ev_{(p)} : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$

"12 has a zero of order 2 at (2) and a zero of order 1 at (3)"

Def: If R is a ring, denote (spectrum of R)

$$\operatorname{Spec} R = \{\text{prime ideals } P \subseteq R\}$$

We've seen $\operatorname{Spec} \mathbb{Z}$, $\operatorname{Spec} \mathbb{C}[x] = \mathbb{A}^1_{\mathbb{C}}$

In an ideal world, we'd like a topology on X s.t.

$R = $ ring of cts. fns. on X

subtle

Def: If $S \subseteq R$, we define:

$$\mathbb{V}(S) = \{P \in \operatorname{Spec} R : S \subseteq P\}$$

Ex: If $R = \mathbb{C}[x]$, $\mathbb{V}(\{x^2+1\}) = \mathbb{V}((x+i)(x-i)) = \underset{\substack{\| \\ \text{roots of } x^2+1}}{\{(x+i), (x-i)\}}$

Def: The Zariski topology on $\operatorname{Spec} R$ is the top. whose closed

Def: The Zariski topology on $\operatorname{Spec} R$ is the top. whose <u>closed</u> sets are
$$\{\mathbb{V}(S) : S \subseteq R\}$$

Prop: This is indeed a topology

Pf: $\emptyset = \mathbb{V}(1)$, $\operatorname{Spec} R = \mathbb{V}(0)$

$$\bigcap_{i\in I} \mathbb{V}(S_i) = \mathbb{V}\Big(\sum_{i\in I} S_i\Big)$$ (exercise)

$\leftarrow$ finite sums $\sum_{i\in I} s_i$, where $s_i \in S_i$

$$\mathbb{V}(S_1) \cup \mathbb{V}(S_2) = \mathbb{V}(S_1 S_2)$$

"$\{s_1 s_2 : s_i \in S_i\}$" $\square$

Observe that $\mathbb{V}(S) = \mathbb{V}(\langle S\rangle)$ (since prime ideals are ideals and $\langle S\rangle$ = the smallest ideal containing S), so it suffices to consider $\mathbb{V}(I)$ for $I \subseteq R$; this goes nicely with the above Prop., as $\langle I_1 I_2\rangle$ and $\sum_{i\in J} I_i$ are also ideals

Prop: If $f: R \to S$ is a ring hom., $f^*: \operatorname{Spec} S \to \operatorname{Spec} R$ is a cts. map.

Pf: First, note that $f^{-1}(P)$ (for $P \subseteq S$) is prime in R

$$ab \in f^{-1}(P) \iff f(ab) \in P \iff f(a)f(b) \in P$$
$$\Updownarrow$$
$f(a) \in P$ or $f(b) \in P$

$$\ddot\smile$$

$a \in f^{-1}(P)$ and $\cancel{\ldots} \implies f(a) \in P$ or $f(b) \in P$

So, $P \longmapsto f^{-1}(P)$ defines a map $\mathrm{Spec}\, S \longrightarrow \mathrm{Spec}\, R$.

For continuity, note that $(f^{\#})^{-1}(\mathbb{V}(S)) = \mathbb{V}(f^{-1}(S))$. $\square$

<u>Note</u>: We have defined a functor $\mathrm{Rings}^{op} \longrightarrow \mathrm{Top}$; eventually, we'll upgrade this to an equivalence $\mathrm{Rings}^{op} \xrightarrow{\sim} \{\text{affine schemes}\}$

$$\left[\begin{array}{c}\begin{tikzcd} \rho \arrow[r] & (0) \arrow[r, mapsto] & \mathbb{C}[x] \arrow[r, "id"'] & \mathbb{C}[x] \arrow[r] \arrow[d] & 0 \\ & & & \mathbb{C}(x) & \end{tikzcd}\end{array}\right]$$

Recall that a sheaf has 3 pieces of data: set, topology & (sheaf-specific-stuff)
$\underbrace{}_{\text{fns. on opens}}$

We can define sheaves just from their values on a <u>basis</u> for our topology.

Instead of defining $\mathcal{F}(U)$ is for any open, can just look at nice opens.

opens.

Prop: The sets $D(f) = \operatorname{Spec} R \setminus \mathbb{V}(f)$ for $f \in R$ form a basis for the Zariski top. $\square$

Pf: $\mathbb{V}(S) = \operatorname{Spec} R \setminus \bigcup_{f \in S} D(f)$.

If we have $\underset{\operatorname{Spec} S}{\overset{\shortparallel}{U}} \hookrightarrow \underset{\operatorname{Spec} R}{\overset{\shortparallel}{X}}$, assume that $X = \operatorname{Spec} R$, $U = \operatorname{Spec} S$

$$R \longrightarrow S$$

If we think about $R = \mathbb{C}[X]$, $U = D(x)$, then $\frac{1}{x}$ is a good candidate for function defined on U, but on $\mathbb{C}$.

$\mathbb{C}[x, x^{-1}]$

Def: If $A \subseteq R$ is a mult. subset, we define

$R[A^{-1}] =$ "fractions $\frac{r}{a}$ for $r \in R, a \in A$"

(formally: recall the construction of $\mathbb{Q}$ from $\mathbb{Z}$

(r, a), can define $(r, a) \sim (r', a')$ iff

$s(ra' - r'a) = 0$ for some $s \in A$

$+$ and $\cdot$ defined via fraction arithmetic

How are $\operatorname{Spec} R$ and $\operatorname{Spec} R[A^{-1}]$ related?

Prop: $\operatorname{Spec} R[A^{-1}] = \{ \overset{\not{\subseteq} R[A^{-1}]}{P} \in \operatorname{Spec} R \text{ s.t. } P \cap A = \emptyset \}$

Pf: Have a map $R \longrightarrow R[A^{-1}]$, gives a map $\operatorname{Spec} R[A^{-1}] \to \operatorname{Spec} R$
$\qquad\qquad\qquad\qquad\quad r$

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$$m \longmapsto \frac{m}{1}$$

If $P \cap A \neq \emptyset$, then $PR[A^{-1}] = R[A^{-1}]$, so no P.

Observation: If $P \subseteq R$ is prime, $R \setminus P$ is mult. closed $\Rightarrow$ can localize at $R \setminus P$!

$\leadsto$ open subsets of Zariski top.

Finally, we can define our sheaf on $\operatorname{Spec} R$ by:

$$\mathcal{F}(D(f)) = R[f^{-1}] = R[\{1, f, f^2, \dots\}^{-1}]$$

" $\frac{27}{4}$ has a zero of order 3 at 3 & pole of order 2 at 2"

$27 \times 4^{-1} \pmod{p}$ if $p \neq 2$

Final note:

What does quotienting R by an ideal do on Spec?

$$R \twoheadrightarrow R/I \quad \leadsto \quad \operatorname{Spec} R/I \hookrightarrow \operatorname{Spec} R$$

Prop: $\operatorname{Spec}(R/I) \longleftrightarrow \{P \in \operatorname{Spec} R : I \subseteq P\}$

Pf: exercise.

$\mathbb{A}^n_{\mathbb{C}}$, $\mathbb{V}(f)$ is a hypersurface in $\mathbb{C}^n$

$$\operatorname{Spec} \frac{\mathbb{C}[x_1, \dots, x_n]}{(f)}$$

$$\mathrm{Spec}\ \frac{k[x_1,\dots,x_n]}{(f)}$$