Simplicially enriched cats

Defn A monoidal structure on a category V is

- $-\otimes- : V \times V \to V$
- A unit $\mathbb{1} \in V$ & $\eta_\ell : X \to \mathbb{1} \otimes X$, $\eta_r : X \to X \otimes \mathbb{1}$ natural isomorphisms.
- $\alpha : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ natural iso

triangle axiom

$$\begin{tikzcd}
A \otimes (\mathbb{1} \otimes B) \arrow[rr] \arrow[dr] & & (A \otimes \mathbb{1}) \otimes B \arrow[dl] \\
& A \otimes B &
\end{tikzcd}$$

commutes

pentagon axiom

$$\begin{tikzcd}
A \otimes (B \otimes (C \otimes D)) \arrow[r] \arrow[d] & (A \otimes B) \otimes (C \otimes D) \arrow[d] \\
A \otimes ((B \otimes C) \otimes D) \arrow[dr] & ((A \otimes B) \otimes C) \otimes D \arrow[d] \\
& (A \otimes (B \otimes C)) \otimes D
\end{tikzcd}$$

Thm (Maclane)

All diagrams commute (at least involving associators and unitors).
In particular, higher associativity for free.

Ex $\times$ in any category admitting finite products. Here the unit is the terminal object.

- $(Set, \times, \{*\})$
- $(Cat, \times, [0])$
- $(sSet, \times, \Delta^0)$

Defn lax monoidal functors and nat transformations are functors and nat transf. that respect $\otimes$, $\mathbb{1}$, associativity, unitality.

(Lax mon functor) $F: V \to W$

- map $\mathbb{1}_W \to F(\mathbb{1}_V)$
- nat map $FX \otimes_W FY \to F(X \otimes_V Y)$

* not isos.

* direction matters

other direction gives oplax functors

Lax mon. nat. transf. $\tau : F \Rightarrow G$

$$\begin{tikzcd} FX \otimes FY \arrow[r] \arrow[d] & F(X \otimes Y) \arrow[d] \\ GX \otimes GY \arrow[r] & G(X \otimes Y) \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{1} \arrow[d] \arrow[dr] & \\ F(\mathbb{1}) \arrow[r] & G(\mathbb{1}) \end{tikzcd}$$

commute

Remark

$\mathrm{MonCat}^{\text{lax}}$ is a 2-category where V, W the hom-category is $\mathrm{Fun}^{\text{lax}}(V, W)$

Defn Let $(V, \otimes, \mathbb{1})$ be a monoidal cat. A V-enriched category $\mathcal{C}$ is a collection of objects and $\forall x, y \in \mathcal{C}$ a $\mathrm{Hom}_{\mathcal{C}}(x,y) \in V$ with composition

$$\mathrm{Hom}_{\mathcal{C}}(x,y) \otimes \mathrm{Hom}_{\mathcal{C}}(y,z) \to \mathrm{Hom}_{\mathcal{C}}(x,z)$$

and an identity id_X

$$\mathbb{1} \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, X)$$

satisfying associativity & unitality.

Examples

- A category enriched in $(\mathrm{Set}, \times)$ is a category
- A cat-enriched category is a <u>strict</u> 2-category.

A (strict) 2-category $\mathcal{C}$

comp here is vertical comp

- A collection of objects
- $\forall x, y$ a category $\mathcal{C}(x, y)$
- an $id_X \in \mathcal{C}(X, X)$ picking out id_X with id_{id_X} 2-morphism
- Composition functors (horizontal composition)

$$\mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C)$$

- which is (strictly) unital & associative

Cat is a 2-cat

$$\begin{tikzcd} A \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow"] \arrow[r, phantom, shift right=3, "\Downarrow"] & B \end{tikzcd} \quad \text{gives} \quad \begin{tikzcd} A \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow"] & B \arrow[r, bend left, "H"] \arrow[r, bend right, "J"'] \arrow[r, phantom, "\Downarrow"] & C \end{tikzcd} \longrightarrow \begin{tikzcd} A \arrow[rr, bend left, "H \circ F"] \arrow[rr, bend right, "J \circ G"'] \arrow[rr, phantom, "\Downarrow"] & & C \end{tikzcd}$$

$$\begin{tikzcd} & (C(C,D) \times C(B,C)) \times C(A,B) \arrow[dl] \arrow[dr, Rightarrow] & \\ C(B,D) \times C(A,B) \arrow[dr] \arrow[rr, Rightarrow, "\cong"'] & & C(C,D) \times C(A,C) \arrow[dl] \\ & C(A,D) & \end{tikzcd}$$

if nonstrict

i.e. $(f \circ g) \circ h \xrightarrow{\text{2-isomorphism}} f \circ (g \circ h)$

Commutes (if strict)

or up to nat transf (if not)

<u>Ex</u> A Cat-enriched category with one object is a strictly associative monoidal category.

<u>Def</u> A V-enriched functor $f: C \to D$ is $x \longmapsto f(x)$

a mor $\mathrm{Hom}_C(x,y) \longrightarrow \mathrm{Hom}_C(f(x), f(y))$

$$\begin{tikzcd}
\mathrm{Hom}(x,y) \otimes \mathrm{Hom}(y,z) \arrow[r] \arrow[d] & \mathrm{Hom}(x,z) \arrow[d] \\
\mathrm{Hom}(fx,fy) \otimes \mathrm{Hom}(fy,fz) \arrow[r] & \mathrm{Hom}(fx,fz)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "id_x"] \arrow[d, "id_x"'] & \mathrm{Hom}(x,x) \arrow[dl] \\
\mathrm{Hom}(fx,fx) &
\end{tikzcd}$$

Def Cat_V is the category of V-enriched categories & V-enriched functors.

Lemma

A lax monoidal functor $\phi : V \to V'$ gives a functor $\phi_* : Cat_V \to Cat_{V'}$

this determines a 2-functor

$$MonCat^{lax} \longrightarrow Cat$$

Pf

2-functor

$$\mathrm{MonCat}^{lax} \longrightarrow \mathrm{Cat}\ ;\ V \longmapsto \mathrm{Cat}_V$$

$$\phi : V \to V'$$

$$\phi_* : \mathrm{Cat}_V \longrightarrow \mathrm{Cat}_{V'}$$

given $\mathcal{C} \in \mathrm{Cat}_V$ need to build a V' enriched $\phi_*(\mathcal{C})$

objects are the same as $\mathcal{C}$

and $\mathrm{Hom}_{\phi_*(\mathcal{C})}(x,y) = \phi(\mathrm{Hom}_{\mathcal{C}}(x,y))$

$$C \xrightarrow{F} C' \qquad \mathrm{Hom}(x,y) \xrightarrow{F} \mathrm{Hom}(Fx, Fy)$$

$$\phi_* \mathcal{C} \longrightarrow \phi_* \mathcal{C}' \qquad \phi(\mathrm{Hom}_{\mathcal{C}}(x,y)) \xrightarrow{\phi F} \phi(\mathrm{Hom}(Fx, Fy))$$

2-morphisms

a lax mon nattrans

$$\tau : \phi \to \psi$$

is sent to $\tau_* : \phi_*(\mathcal{C}) \longrightarrow \psi_*(\mathcal{C})$

identity on objects

$$\tau_* : \phi(\mathrm{Hom}_{\mathcal{C}}(x,y)) \to \psi(\mathrm{Hom}_{\mathcal{C}}(x,y))$$

is $\tau_{\mathrm{Hom}(X,Y)}$. $\square$

It follows that monoidal adjunctions determine adjunctions of enriched categories.

V monoidal cat. & $\mathcal{C}$ is (locally small) V-enriched then its underlying cat $u\mathcal{C}$ is given by

$$\mathrm{Hom}(\mathbb{1}, -) : V \longrightarrow \mathrm{Set}$$

$$u\mathcal{C} = \mathrm{Hom}_V(\mathbb{1}, -)_*(\mathcal{C}) \in \mathrm{Cat}$$

Def A simplicial category is $\mathcal{C} \in \mathrm{Cat}_{sSet} = \mathrm{Cat}_\Delta$

Ex if $\mathcal{C} \in \mathrm{Cat}_\Delta$ then

$u\mathcal{C}$ has the same obs as $\mathcal{C}$ and $\mathrm{Hom}(x,y) = (\mathrm{Hom}_\Delta(x,y))_0$

Lemma

there is a fully faithful embedding

$$\mathrm{Cat}_\Delta \xrightarrow{i} \mathrm{Fun}(\Delta^{op}, \mathrm{Cat})$$

arises from

$$ev_n : s\mathrm{Set} \to \mathrm{Set}$$

$$X \mapsto X_n$$

lift to $(ev_n)_* \mathrm{Cat}_\Delta \to \mathrm{Cat}$

given $\mathcal{C} \in \mathrm{Cat}_\Delta$

$$\big(i(\mathcal{C})\big)_n = (ev_n)_*(\mathcal{C})$$

Cor

Cat_Δ is bicomplete

Proof Given an I-shaped diagram in Cat_Δ, we get an I-shaped diagram in $\mathrm{Fun}(\Delta^{op}, \mathrm{Cat})$

we get $\mathrm{colim}\, \mathcal{C}_i$

$$\mathrm{ob}(\mathrm{colim}\, \mathcal{C}_i) \cong \mathrm{colim}(\mathrm{ob}(\mathcal{C}_i))$$

$\cap$ sSet

$$c : \mathrm{Set} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} s\mathrm{Set} : ev_0$$