

Simplicially enriched cats

Defn A monoidal structure on a category V is

- $- \otimes - : V \times V \rightarrow V$
- A unit $\mathbb{1} \in V$ & $\eta_L : X \rightarrow \mathbb{1} \otimes X$
 $\eta_R : X \rightarrow X \otimes \mathbb{1}$ natural isomorphisms.
- $\alpha : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$
natural iso

$$A \otimes (\mathbb{1} \otimes B) \rightarrow (A \otimes \mathbb{1}) \otimes B$$

triangle
axiom

$$\searrow \quad \swarrow$$

$$A \otimes B$$

commutes

$$A \otimes (B \otimes (C \otimes D)) \rightarrow (A \otimes B) \otimes (C \otimes D)$$

pentagon
axiom

$$\downarrow$$

$$A \otimes ((B \otimes C) \otimes D)$$

$$\downarrow$$

$$((A \otimes B) \otimes C) \otimes D$$

$$\searrow \quad \swarrow$$

$$A \otimes (B \otimes C) \otimes D$$

Thm (MacLane)

All diagrams commute (at least involving associators and unitors).

In particular, higher associativity for free.

\prod X in any category admitting finite products. Here the unit is the terminal object.

- $(\text{Set}, X, \{*\})$
- $(\text{Cat}, X, [0])$
- $(\text{sSet}, X, \Delta^0)$

Defn lax monoidal functors and natural transformations are functors and nat trans that respect $(\otimes, \mathbb{1}$, associativity, unitality).

(Lax mon functor) $F: V \rightarrow W$

map $\mathbb{1}_W \rightarrow F(\mathbb{1}_V)$

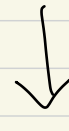
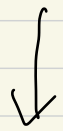
• nat map $F(X \otimes_V Y) \rightarrow F(X) \otimes_W F(Y)$

* Not isos.

* direction matters
other direction gives oplax functors

lax mon. nat. transf. $\tau: F \Rightarrow G$

$$FX \otimes FY \rightarrow F(X \otimes Y)$$



$$GX \otimes GY \rightarrow G(X \otimes Y)$$

$$\begin{array}{ccc} \mathbb{I} & & \\ \downarrow & \searrow & \\ F(\mathbb{I}) & \longrightarrow & G(\mathbb{I}) \end{array} \quad \underline{\text{commute}}$$

Remark

Mon Cat^{lax} is a 2-category
where V, W the hom-category
is $\text{Fun}^{\text{lax}}(V, W)$

Defn Let (V, \otimes, \mathbb{I}) be a
monoidal cat. A V -enriched
category \mathcal{C} is a collection of
objects and $\forall x, y \in \mathcal{C}$ a
 $\text{Hom}_{\mathcal{C}}(x, y) \in V$ with composition
 $\text{Hom}_{\mathcal{C}}(x, y) \otimes \text{Hom}_{\mathcal{C}}(y, z) \rightarrow \text{Hom}_{\mathcal{C}}(x, z)$

and an identity id_x

$$I \longrightarrow \text{Hom}_e(x, x)$$

Satisfying associativity & unitality.

Examples

- A category enriched in (Set, \times) is a category
- A cat-enriched category is a strict 2-category.

A (strict) 2-category \mathcal{C}

• A collection of objects here is a vertical comp

• $\forall x, y$ a category $\mathcal{C}(x, y)$

• an $id_x \in \mathcal{C}(x, x)$ picking out

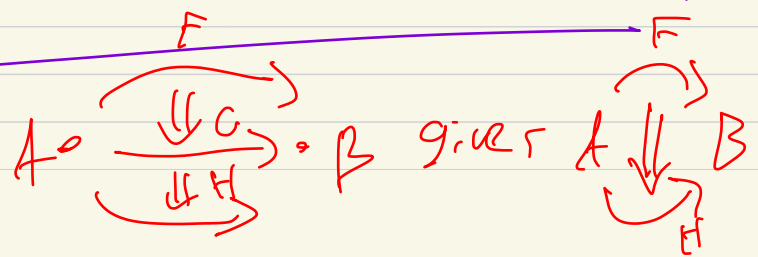
id_x with id_{id_x} 2-morphism

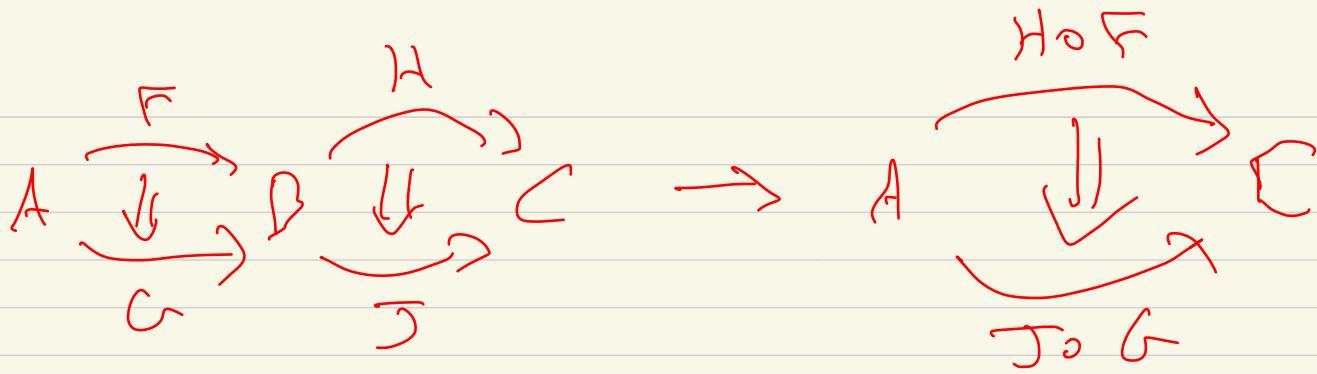
• Composition functors (horizontal composition)

$$\mathcal{C}(B, C) \times \mathcal{C}(A, B) \rightarrow \mathcal{C}(A, C)$$

• which is (strictly) unital & associative

Cat is a 2-cat





$$(C(C, D) \times (C(B, C))) \times (C(A, B))$$

$$C(B, D) \times C(A, B) \xrightarrow{\cong} C(C, D) \times C(A, C)$$

if \cong is strict

$$C(A, D)$$

i.e. 2-isomorphism
 $(f \circ g) \circ h \rightarrow f \circ (g \circ h)$

commutes (if strict)

or up to nat transf (if not)

Ex A Cat-enriched category with one object is a strictly associative monoidal category.

Def A V -enriched functor

$$f: C \rightarrow D \text{ is } x \mapsto f(x)$$

$$\text{a nat } \text{Hom}_C(x, y) \rightarrow \text{Hom}_D(f(x), f(y))$$

$$\text{Hom}(x, y) \otimes \text{Hom}(y, z) \rightarrow \text{Hom}(x, z)$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$\text{Hom}(f_x, f_y) \otimes \text{Hom}(f_y, f_z) \rightarrow \text{Hom}(f_x, f_z)$$

$$\mathbb{1} \xrightarrow{\text{id}_x} \text{Hom}(x, x)$$

$$\text{id}_x \downarrow \qquad \checkmark$$

$$\text{Hom}(f_x, f_x)$$

Def Cat_V is the category of V -enriched categories & V -enriched functors.

Lemma

A lax monoidal functor

$\phi: V \rightarrow V'$ gives a functor

$\phi_*: \text{Cat}_V \rightarrow \text{Cat}_{V'}$

this determines a 2-functor

$$\text{MonCat}^{\text{lax}} \rightarrow \text{Cat}$$

Pf

2-functor

$$\text{MonCat}^{\text{Lat}} \longrightarrow (\text{Cat}; V \mapsto \text{Cat}_V)$$

$$\phi : V \longrightarrow V'$$

$$\phi_{\#} : (\text{Cat}_V \longrightarrow \text{Cat}_{V'})$$

given $C \in \text{Cat}_V$ need to build
a V' enriched $\phi_{\#}(C)$

objects are the same as C

$$\text{and } \text{Hom}_{\phi_{\#}(C)}(x, y) = \phi(\text{Hom}_C(x, y))$$

$$C \xrightarrow{F} C' \quad \text{Hom}(x, y) \xrightarrow{F} \text{Hom}(Fx, Fy)$$

$$\phi_{\#} C \longrightarrow \phi_{\#} C' \quad \phi(\text{Hom}_C(x, y)) \xrightarrow{\phi F} \phi(\text{Hom}_{C'}(Fx, Fy))$$

2-morphisms

a lax mon natural

$$\tau : \phi \rightarrow \psi$$

$$\text{is sent to } \tau_{\#} : \phi_{\#}(C) \rightarrow \psi_{\#}(C)$$

identity on objects

$$\tau_{\#} : \phi(\text{Hom}_C(x, y)) \rightarrow \psi(\text{Hom}_C(x, y))$$

is $\tau_{\text{Hom}(x,y)}$. \square

It follows that monoidal adjunctions determine adjunctions of enriched categories.

V monoidal $\text{Cat}^{V, \text{locally small}}$ \mathcal{C} is V -enriched then its underlying Cat $u\mathcal{C}$ is given by

$$\text{Hom}(\mathbb{1}, -) : V \rightarrow \text{Set}$$

$$u\mathcal{C} = \text{Hom}_V(\mathbb{1}, -)_* (\mathcal{C}) \in \text{Cat}$$

Def A simplicial category

$$\text{is } \mathcal{C} \in \text{Cat}_{\text{Set}} = \text{Cat}_{\Delta}$$

Ex if $\mathcal{C} \in \text{Cat}_{\Delta}$ then

$u\mathcal{C}$ has the same obj as \mathcal{C}

$$\text{and } \text{Hom}(x,y) = (\text{Hom}_{\Delta}(x,y))_0$$

Lemma

there is a fully faithful embedding

$$\text{Cat } \Delta \xrightarrow{i} \text{Fun}(\Delta^{\text{op}}, \text{Cat})$$

arises from

$$\text{ev}_n : \text{Set} \rightarrow \text{Set}$$

$$X \mapsto X_n$$

$$\text{lift to } (\text{ev}_n)_* \text{Cat}_\Delta \rightarrow \text{Cat}$$

given $C \in \text{Cat}_\Delta$

$$(i(C))_n = (\text{ev}_n)_*(C)$$

Cor

Cat_Δ is bicomplete

Proof Given an I -shaped diagram

in Cat_Δ , we get an I -shaped

diagram in $\text{Fun}(\Delta^{\text{op}}, \text{Cat})$

we get $\text{colim } C_i$

$$\text{ob}(\text{colim } C_i) \cong \text{colim}(\text{ob}(C_i))$$

$$C : \text{Set} \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \perp \\ \xrightarrow{\quad} \end{array} \text{Set} \text{ via } \text{ev}_n$$

$$\text{Set}$$