Sheares

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Eisenboaties

Motivation: differentiable fns

Let X be a real diff. manifold.

We can understand X by studying set/ring of differentiable functions $\mathcal{O}(X)$ on it.

Properties

- Given open subsets $U \subset V$, $\mathcal{O}(V) \xrightarrow{res_{V,U}} \mathcal{O}(U)$
- Given open $U \subset V \subset W$,

$$\begin{tikzcd} \mathcal{O}(W) \arrow[r] \arrow[dr] & \mathcal{O}(V) \arrow[d] \\ & \mathcal{O}(U) \end{tikzcd}$$ commutes.

- given $f, g \in \mathcal{O}(U)$ and $U = \bigcup_i U_i$ and $f|_{U_i} = g|_{U_i}$, then $f = g$ (identity)

- Given $f_i \in \mathcal{O}(U_i)$ and $U = \bigcup U_i$ s.t. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ then

  $\exists f \in \mathcal{O}(U)$ s.t. $f|_{U_i} = f_i$ (gluing)

These are the properties of a sheaf (of sets) on X.

Stalks and germs

A stalk at $p \in X$ is

$$\mathcal{O}_p = \{ (f, U) : p \in U, f \in \mathcal{O}(U) \} / \sim$$

where $(f,U) \sim (g,V) \Leftrightarrow \exists \underset{p}{\overset{\Downarrow}{W}} \subseteq U \cap V$ s.t. $f|_W = g|_W$

An element of $\mathcal{O}_p$ is called a germ.

Notice $\mathcal{O}_p$ is a ring.

$(f,U), (g,V)$ $(f+g, U \cap V)$ same for $\cdot$

in fact, it is a local ring.

Notice $\mathcal{F}(U) \to \mathcal{F}_p$

$\mathcal{O}_p$ has maximal ideal

$m_p = \{$ germs that vanish at $p \}$.

It is maximal since

$$\mathcal{O}_p / m_p \cong \mathbb{R} \text{ via}$$

$$[f] \longmapsto f(p)$$

The maximal ideal is unique since any $\underline{f} \in \mathcal{O}_p \setminus m_p$ is invertible with inverse

$$(1/f, W) \quad W \text{ small enough}.$$

Note we can identify $\underline{f(p)}$ by the image of f in $\mathcal{O}_p / m_p$.

Presheaves, generally

A presheaf on a small category C valued in a category D is a functor

$$F : C^{op} \longrightarrow D .$$

Today: C = Open(X) (morphisms are inclusions)

D = Set (unless otherwise specified)
Abelian groups, Rings

Generally, when advancing to sheaves we need C to be a site.

Def Given $p \in X$ and a presheaf $\mathcal{F}$ on X, we define the stalk $\mathcal{F}_p$ at $p \in X$ as

$$\mathcal{F}_p = \operatorname*{colim}_{U \ni p} \mathcal{F}(U)$$

We define the germ of $f \in \mathcal{F}(U)$ at p as the image of $\mathcal{F}(U) \to \mathcal{F}_p$ of f.

Given a diagram in a category. A cocone of

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] \arrow[ddr, bend right] & Y \arrow[r] \arrow[d] \arrow[dd, bend right] & Z \arrow[dl] \arrow[ddl, bend left] \\
& \text{colimit} \arrow[d, "\exists!"] & \\
& c &
\end{tikzcd}$$

is an object c.

The colimit is the "initial cocone"

Sheaves

Def A sheaf is a presheaf $\mathcal{F}$ satisfying

- (identity) If $U = \bigcup_i U_i$ and $f_1, f_2 \in \mathcal{F}(U)$ are such that $\mathrm{res}_{U_i, U_i} f_1 = \mathrm{res}_{U_i, U_i} f_2$ then $f_1 = f_2$.
- (gluability) If $U = \bigcup_i U_i$ and $f_i \in \mathcal{F}(U_i)$ is such that $\mathrm{res}_{U_i, U_i \cap U_j} f_i = \mathrm{res}_{U_j, U_i \cap U_j} f_j$ then $\exists f \in \mathcal{F}(U)$ with $\mathrm{res}_{U_i} f = f_i$.

Note in particular, $\mathcal{F}(\emptyset) = \{*\}$

$F(\emptyset) = \{*\}$

$\emptyset = \bigcup_{\emptyset} \emptyset$ by glubility

for every $\emptyset$ we have a $f \in F(\emptyset)$

$f, g \in F(\emptyset)$ s.t. $\emptyset$ then $f = g$

(identity)

Def 2 A sheaf is a presheaf F

s.t. $\forall\, U = \bigcup U_i$,

$$F(U) \to \prod F(U_i) \underset{F(\iota_2)}{\overset{F(\iota_1)}{\rightrightarrows}} \prod F(U_i \cap U_j)$$

is an equalizer

induced by $\boxed{\begin{array}{l} \iota_1 : U_i \cap U_j \hookrightarrow U_i \\ \iota_2 : U_i \cap U_j \hookrightarrow U_j \end{array}}$

$$f \in F(U_i) \longrightarrow F(U_j \cap U_k)$$

$$g \in (F(U_j)) \longrightarrow$$

$$F(\iota_1) : (f_1, f_2, f_3) \longrightarrow \begin{pmatrix} f_1|_{U_1 \cap U_1} & f_1|_{U_1 \cap U_2} & f_1|_{U_1 \cap U_3} \\ f_2 & & \\ & & \end{pmatrix}$$

$$F(\iota_2) = F(\iota_1)^T$$

$f, g \longmapsto$ same thing

$$\mathcal{F}(U) \to \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \cap U_j)$$

(identity) $f, g \in \mathcal{F}(U)$

s.t. $f_i = g_i$ then $f = g$

(equalizer is inj)

(gluing) if $(f_i) \in \prod \mathcal{F}(U_i)$ equalized by maps but no $f \in \mathcal{F}(U)$

then $\mathcal{F}(U) \cup \{(f_i)\}$ would not have a unique map to $\mathcal{F}(U)$ so not equalizer.

Def Given a sheaf $\mathcal{F}$ on X and open $U \subseteq X$, $\mathcal{F}|_U$ is the sheaf on U given by $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for $V \subseteq U$.

Example Fix $p \in X$ and a set S. The skyscraper sheaf is the sheaf

$$i_p S(U) = \begin{cases} S & \text{if } p \in U \\ \cancel{\emptyset} & \text{otherwise} \end{cases}$$

Ex: check this is a sheaf!

Example The constant presheaf for a fixed set S is the presheaf

$$\underline{S}(U) = S \quad \cancel{\text{and } \underline{S}(\emptyset) = *}$$

Warning This is not a sheaf!

Let $X = \{1, 2\}^{\delta}$ $\quad S = \{x, y\}$

$X = \{1\} \cup \{2\}$

$x \in \underline{S}\{1\}$ $\quad$ by gluing I should find
$y \in \underline{S}\{2\}$ $\quad$ $f \in \underline{S}\{1, 2\}$:(

Def The constant sheaf for a set S is given by

$$\mathcal{F}(U) = \{\text{cts maps } U \to S^{\delta} \text{ that are locally constant}\}$$

Example The sheaf on X of sections of a cts map $\pi: Y \to X$

$$\mathcal{F}(U) = \{ p : U \to Y : \pi \circ p = \mathrm{id}|_U \}$$

Example Given $\pi : X \to Y$ and a (pre)sheaf $\mathcal{F}$ on X, then the pushforward (pre)sheaf $\pi_* \mathcal{F}$ on Y is given by

$$\pi_* \mathcal{F}(V) = \mathcal{F}(\pi^{-1}(V))$$

π_* is a functor $Sh(X) \to Sh(Y)$

The category of (pre)sheaves

The category $PSh(X)$ valued in Set is $\mathrm{Fun}(\mathrm{Open}(X)^{op}, \mathrm{Set})$

- Stalks are functors

$$\phi_p : \mathrm{Set}_X \longrightarrow \mathrm{Set}$$

because colim is functorial

The category of sheaves $Sh(X)$ is just the <u>full</u> subcategory $Sh(X) \subset PSh(X)$

Internal $\underline{\mathrm{Hom}}$ for $Sh(X)$

Def $\mathcal{F}, \mathcal{G} \in Sh(X)$

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{Sh}(\mathcal{F}|_U, \mathcal{G}|_U)$$

$$\neq \mathrm{Hom}_{Set}(\mathcal{F}(U), \mathcal{G}(U))$$

$$\mathcal{F} \times \mathcal{G}(U) = \mathcal{F}(U) \times \mathcal{G}(U) \quad ?$$

The structure sheaf

If $\mathcal{O}_X$ is a sheaf of rings on X then $(X, \mathcal{O}_X)$ is called a ringed space and $\mathcal{O}_X$ is called the structure sheaf

$\mathcal{O}_X$-modules is a sheaf F of abelian groups s.t. $F(U)$ is an $\mathcal{O}_X(U)$-mod

$V \subseteq U$

$$\begin{tikzcd} \mathcal{O}_X(U) \times F(U) \arrow[r, "\circ"] \arrow[d] & F(U) \arrow[d] \\ \mathcal{O}_X(V) \times F(V) \arrow[r, "\circ"] & F(V) \end{tikzcd}$$

commutes

The category of presheaves is abelian

Thm the category of presheaves on Ab (or $\mathcal{O}_X$-mod)

Pf It's additive

$\mathcal{F} \xrightarrow{f,g} \mathcal{G}$

$\mathcal{F} \xrightarrow{f+g} \mathcal{G}$ is defined on open sets

$\mathcal{F}(U) \xrightarrow{f+g} \mathcal{G}(U)$

everything done open-by-open

Aside

sheaves on Ab groups are $\mathbb{Z}$-mod

Category of sheaves on Ab

- Still additive ✓
- kernels ✓
- Cokernels :(

$X = \mathbb{C}$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{f \mapsto e^{2\pi i f}} \mathcal{F} \longrightarrow 0$$

not a sheaf

exact in psh

$\mathcal{O}_X$: holomorphic functions on X

$\mathcal{F}$: presheaf of functions that admit a log

$\mapsto$ locally constant to $\mathbb{Z}$

$e^{2\pi i 0} = 1$

$\mathbb{C}$

z

$$e^{f(z)} = z$$

Checking properties on stalks

Motto: everything can be checked on stalks

$$F(U) \longrightarrow \prod F_p \text{ is injective}$$

↑ determined by germs

Sheafification

$$\begin{tikzcd} Sh(X) \arrow[r, hook, "U"'] \arrow[r, phantom, "\perp"] & PSh(X) \arrow[l, bend right, "sh"'] \end{tikzcd}$$

U is fully faithful
so $Sh(X)$ is a reflective subcat

1) U is a right adj so limits in sh are limits in Psh (ex kernel)
2) $Sh(X)$ admits all colimits that $PSh(X)$ admits found by applying sh

Cor

Sheaves valued in Ab (or $\mathcal{O}_X$-mod) is abelian!

Pf

$$\mathrm{Coh}(\mathcal{F}) = \mathrm{Coh}(F)^{sh}$$

↑ underlying psh

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}^{sh}$$

$$F(U) \to G(U) \to H(U)$$

Sheaves on abelian groups (or $\mathcal{O}_X$-mods) is abelian

Sheaves from a base

In presheaves things are computed open-by-open

In sheaves things are computed stalk-by-stalk.