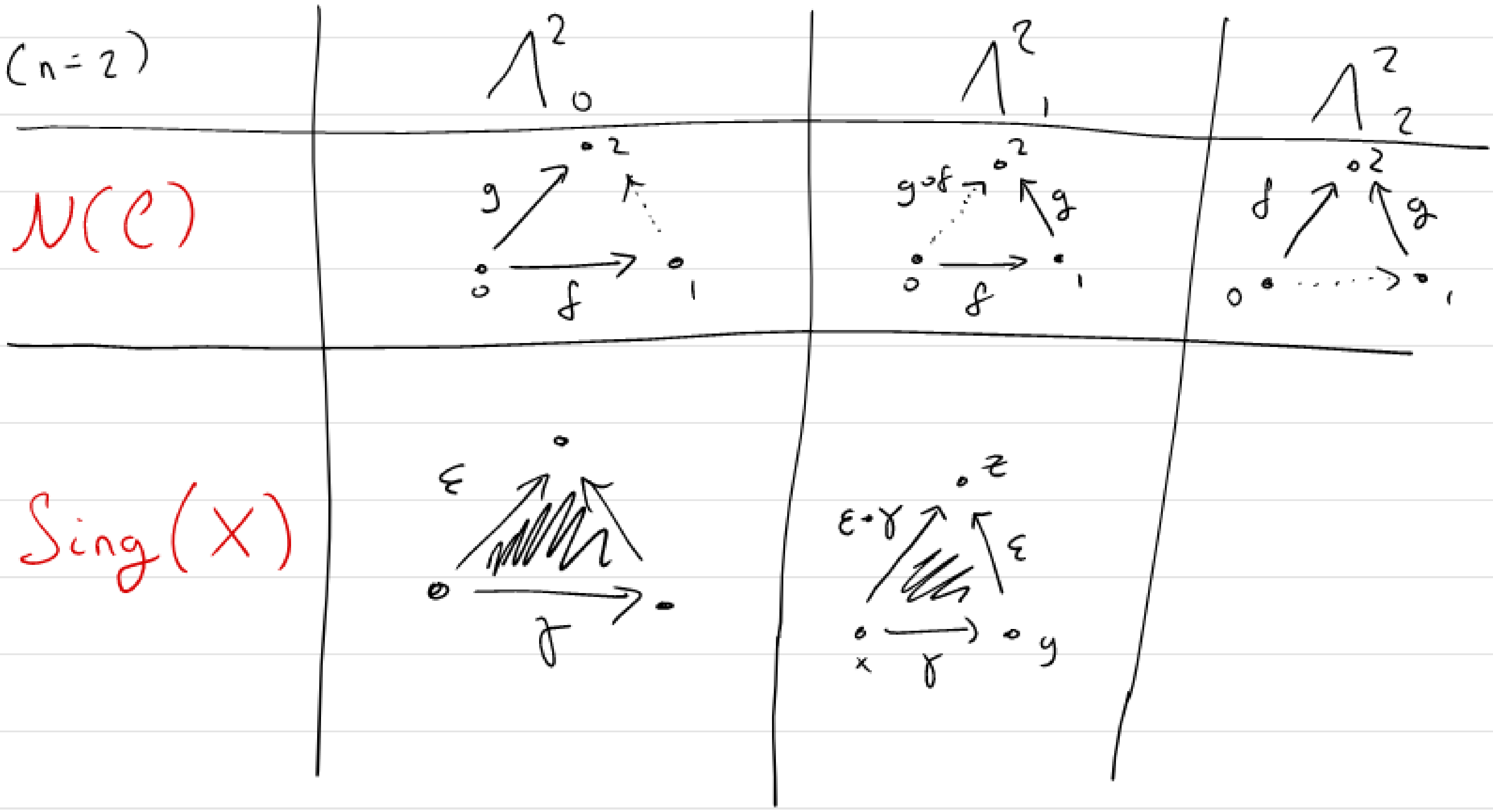
16/3/23

∞-categories

$\boxed{\text{Goal}}$ We want to build a model for ∞-categories. n-simplices should represent n-morphisms (homotopies) between $(n-1)$-morphisms. In particular, we should be able to compose 1-simplices.

Inner horns



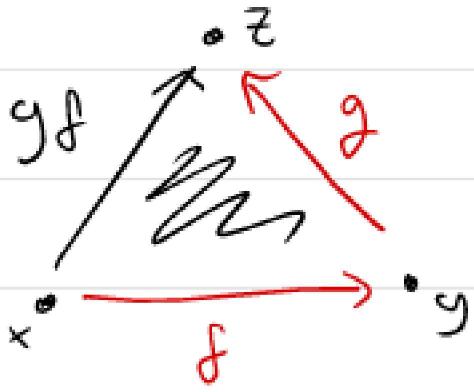
Motto Inner horn lifting is about composability
Outer horn lifting is about homotopies

Why can't we just . . .

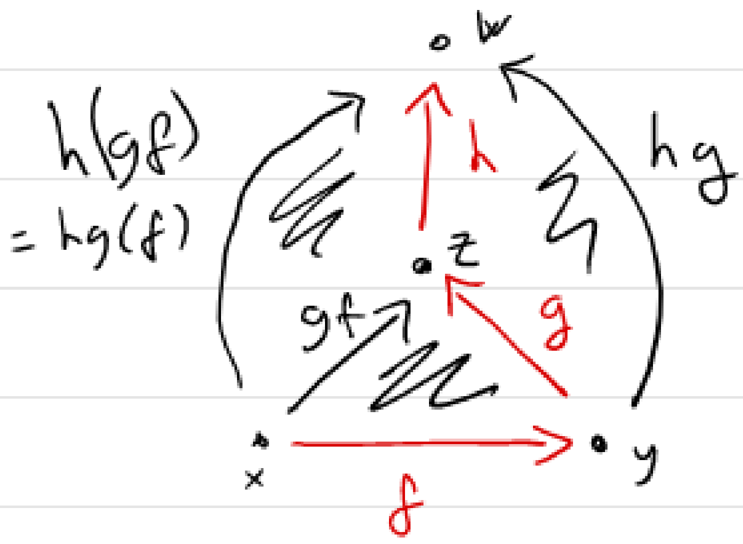
Definition A **composer** is a sSet with extensions along spines

$$I^n \longrightarrow \Delta^n$$

Then we can compose!



And we don't need to worry about associativity!



The homotopy category

If 2-simplices are representing homotopies then we should be able to form the homotopy category hX.

Def Let $X \in sSet$, $f, g \in X_1$.
Then $f \sim g$ if $\exists$ a 2-simplex

$$\begin{tikzcd} & y \\ x \arrow[ur, "g"] \arrow[r, "f"'] & y \arrow[u, "id"'] \end{tikzcd} \quad \not\Rightarrow \quad \begin{tikzcd} & \bullet \\ \bullet \arrow[ur, "f"] \arrow[r, "g"'] & \bullet \arrow[u, "id"'] \end{tikzcd}$$

This is not an equivalence relation

It is reflexive.

$$\begin{tikzcd} & \bullet \\ \bullet \arrow[ur, "f"] \arrow[r, "f"'] & \bullet \arrow[u, "id"'] \end{tikzcd}$$

Not symmetric or transitive

Def Given a sSet X, we construct a category hX generated by X, (call the free composites $f*g$) and subject to

(1) $s_0(x) = \mathrm{id}_x$

(2) if
$$\begin{tikzcd} & \bullet & \\ \bullet \arrow[ur, "h"] \arrow[rr, "f"'] & & \bullet \arrow[ul, "g"'] \end{tikzcd}$$
then $h = g*f$

(3) if $f \sim f'$ then $f*g = f'*g$
$g'*f = g'*f'$

Remarks

- This is functorial $h: sSet \to Cat$
- If X is a composer, then there are no formal composites and any two composites are identified

In particular, if $f \sim f'$ then $[f] = [f']$

$$\begin{tikzcd} & \bullet & \\ \bullet \arrow[ur, "f'"] \arrow[rr, "f"'] & & \bullet \arrow[ul, "id"'] \end{tikzcd} \Rightarrow \boxed{f' \underset{(2)}{=} id * f \underset{(1)}{=} f \quad \text{in } hX}$$

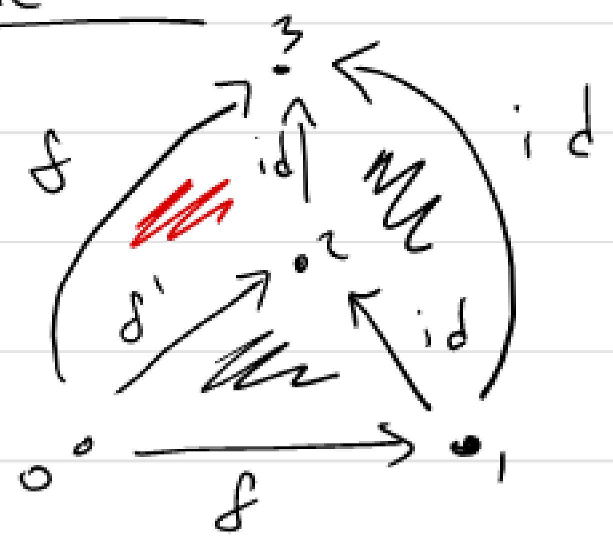
Lemma Let X be a composer with lifts with respect to inner 3-horns.

(a) composites exist.

(b) $\sim$ is an equivalence relation

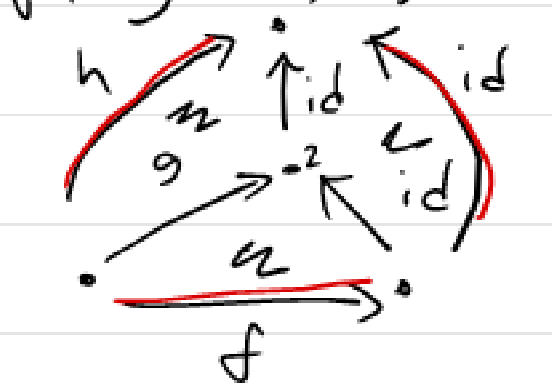
Proof

Symmetric



Transitive

$f \sim g$, $g \sim h$



(c) Any two composites are equivalent under $\sim$

Proof

$$\begin{tikzcd}
& \bullet & \\
& \bullet \arrow[u, "id"] & \\
\bullet \arrow[ur, "h"] \arrow[uur, bend left, "\partial"] \arrow[rr, "f"'] & & \bullet \arrow[ul, "g"'] \arrow[uul, bend right, "g"']
\end{tikzcd}$$

(d)

$$\begin{tikzcd}
& \bullet & \\
\bullet \arrow[ur, "h'"] \arrow[rr, "id"'] & & \bullet \arrow[ul, "h"']
\end{tikzcd} \Rightarrow \quad h \sim h'$$

Proof

In this case we can define $\pi(X)$ whose 1-morphisms are equivalence classes of 1-simplices under $\sim$.

Composition given by lifting along spines is unique by the lemma.

Associativity f, g, h

$$h(gf) = (hg)f$$

$$\begin{tikzcd}
 & \bullet & \\
 & \bullet \arrow[u, "h"] & \\
\bullet \arrow[rr, "f"'] \arrow[ur, "gf"] \arrow[uur, bend left, "h(gf) = (hg)f"] & & \bullet \arrow[ul, "g"'] \arrow[uul, bend right, "hg"']
\end{tikzcd}$$

$h(gf)$ is a composite $(hg) \circ f$

Corollary For composers with inner 3-horn lifts, $\pi(X) \cong h(X)$.

Remark We have only used 2-spines = inner 2-horns lifts and inner 3-horn lifts for this corollary.

Corollary $h(N(\mathcal{C})) \cong \mathcal{C}$

Proof $h(N(\mathcal{C})) \simeq \pi(N(\mathcal{C})) \simeq \mathcal{C}$

Finally...

Definition An ∞-category is a sSet with lifts along all inner horns.

Examples
- $N(\mathcal{C})$
- $\mathrm{Sing}(X)$ or any Kan complex

$= \text{Kan complex}$

Remark In an ∞-category, the space of composites is contractible: π_n^Δ vanishes.
For composers with inner 3-horn lifts we only know π_0^Δ vanishes.
$\hookrightarrow$ by (c) of lemma compositions are unique

$$
\begin{tikzcd}
\mathrm{Comp}_X(f,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \underline{\mathrm{Hom}}(\Delta^2, X) \arrow[d] \\
\Delta^0 \arrow[r, "{(f,g)}"] & \underline{\mathrm{Hom}}(\Lambda^2_1, X)
\end{tikzcd}
$$

by Ex 32 $\pi_0^\Delta \mathrm{Comp}_X(f,g) = 0$

if you have inner 3-horn lifts.

Prop ∞-cats are composers

<u>pf</u> induction

Prop

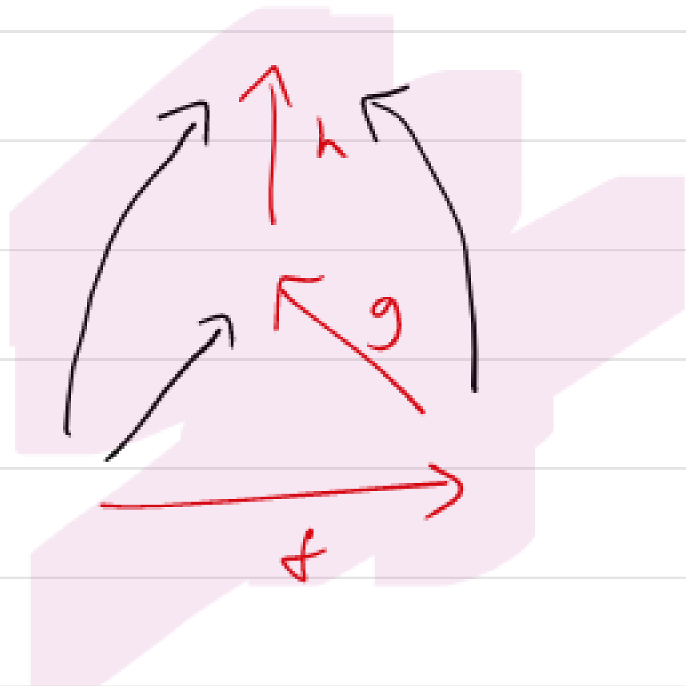
$$\begin{tikzcd} sSet \arrow[r, bend left, "h"] \arrow[r, phantom, "\perp"] & Cat \arrow[l, bend left, "N"] \end{tikzcd}$$

$hN \Rightarrow id$ counit

$hN(C) \simeq C$ by cor

Remark Since the counit $hN \Rightarrow id$ is an isomorphism, N is fully faithful.

<u>Pf</u> Category theory.



Corollary $N(\mathcal{C})$ is 2-coskeletal.

$N(\mathcal{C}) \simeq \mathrm{cosk}_2 N(\mathcal{C})$

i.e. every tetrahedron whose faces commute commutes.

Proof

$$\begin{tikzcd}
\mathrm{Fun}(hX, \mathcal{C}) \arrow[r, "\cong", "\mathrm{adj}"'] \arrow[d, "\cong"', "\mathrm{can}"] & \mathrm{Hom}_{sSet}(X, N(\mathcal{C})) \arrow[d, "\cong"', "\mathrm{com}"] \arrow[dd, bend left=60, "\Rightarrow \cong"] \\
\mathrm{Fun}(h(\mathrm{sk}_2 X), \mathcal{C}) \arrow[r, "\cong \, \mathrm{adj}"'] & \mathrm{Hom}_{sSet}(\mathrm{sk}_2 X, N(\mathcal{C})) \arrow[d, "\cong"', "\mathrm{adj}"] \\
 & \mathrm{Hom}_{sSet}(X, \mathrm{cosk}_2 N(\mathcal{C}))
\end{tikzcd}$$

X is arbitrary, so Yoneda $\Rightarrow$ $N(\mathcal{C}) \simeq \mathrm{cosk}_2(N(\mathcal{C}))$

∞-groupoids

Def an ∞-groupoid is an ∞-cat s.t. $\forall\, \sigma \in X_1$ $h\sigma$ is an iso.

In a category $\mathcal{C}$ we have $\mathcal{C}^{\simeq}$ the maximal subgroupoid.

Def the maximal sub-∞-groupoid of $\mathcal{C}$ is the pullback

$$\begin{tikzcd}
\mathcal{C}^{\simeq} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d, "\text{unit}"] \\
N((h\mathcal{C})^{\simeq}) \arrow[r, "i"] & N(h\mathcal{C})
\end{tikzcd}$$

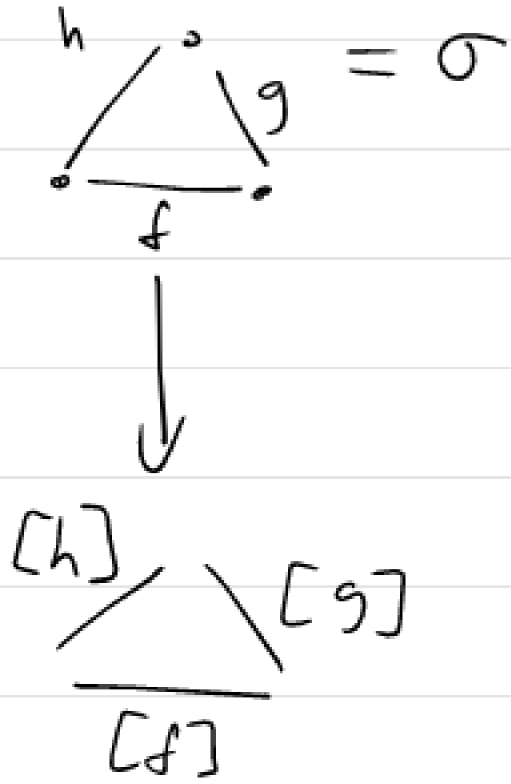
Lemma An n-simplex in $\mathcal{C}$ belongs to $\mathcal{C}^{\simeq}$ $\Leftrightarrow$ every edge is an equivalence.

$\mathcal{C}^{\simeq}$ ← core

Pf

Δ^n

$$\begin{tikzcd} C^{\simeq} \arrow[r] \arrow[d] & C \arrow[d, "\eta"] \\ N(hC^{\simeq}) \arrow[r] & N(hC) \end{tikzcd}$$



Corollary $C^{\simeq}$ is an ∞-groupoid and it is maximal among ∞-groupoids $C' \hookrightarrow C$.

Cat is a 2-category

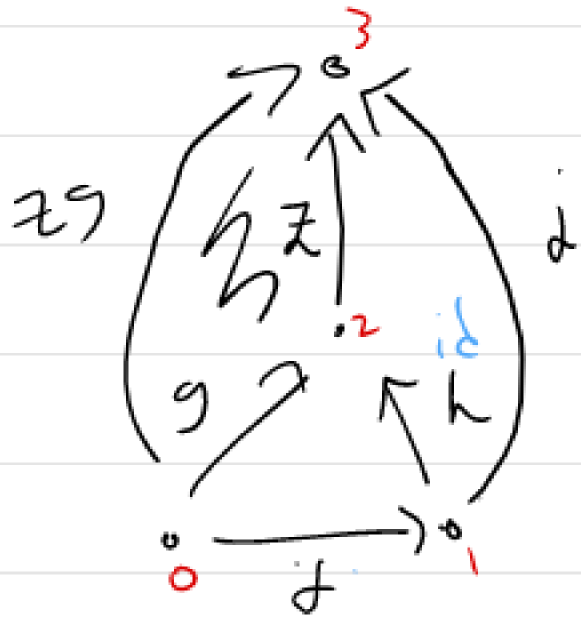
$$\begin{tikzcd}
\bullet \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\alpha"] & \bullet \arrow[r, bend left=50, "H"{name=U2}] \arrow[r, bend right=50, "J"'{name=D2}] \arrow[Rightarrow, from=U2, to=D2] & \bullet
\end{tikzcd}
\longrightarrow
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "H\circ F"{name=U}] \arrow[r, bend right=50, "J\circ G"'{name=D}] \arrow[Rightarrow, from=U, to=D] & \bullet
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[r, bend left=70, "F"{name=U}] \arrow[r, "G"{name=M}] \arrow[r, bend right=70, "H"'{name=D}] \arrow[Rightarrow, from=U, to=M] \arrow[Rightarrow, from=M, to=D] & \bullet
\end{tikzcd}
\longrightarrow
\begin{tikzcd}
\bullet \arrow[r, bend left=60, "F"{name=U}] \arrow[r, bend right=60, "H"'{name=D}] \arrow[Rightarrow, from=U, to=D] & \bullet
\end{tikzcd}$$

Two compositions!

$\mathfrak{m}$ = previous pic

(1)



3-morphism
= 3-simplex

(2)