Integrality etc.

EISENBOATIES 13·04·2023

Recall: A scheme $(X, \mathcal{O}_X)$ is a topological space X equipped with a sheaf of commutative rings $\mathcal{O}_X : \mathrm{Top}(X) \longrightarrow \mathrm{CRing}^{op}$, satisfying [...]

Def [Vak 5.2.4] $(X, \mathcal{O}_X)$ is an integral scheme if

1. The underlying topological space is non-empty, ie $X \neq \emptyset$
2. $\mathcal{O}_X$ is a sheaf of integral domains, ie $\forall U \subseteq X$ open, $\mathcal{O}_X(U)$ is an integral domain

Easy non-example: disconnected schemes $X \sqcup Y$.

$\mathcal{O}(X \sqcup Y) \cong \mathcal{O}(X) \times \mathcal{O}(Y)$, $(1,0) \times (0,1) = (0,0)$

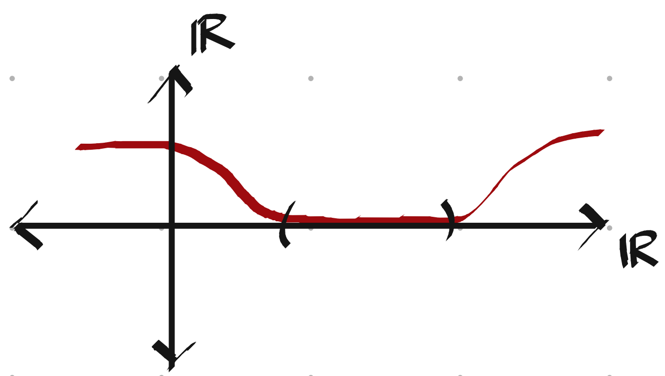
Immediate. If $(X, \mathcal{O}_X)$ is integral, then so are non-empty open subschemes $(U \subseteq X, \mathcal{O}_X|_U)$.

Corollary. All of the following equivalent conditions hold

1. Any two non-empty opens have non-trivial intersection. [ie not disjoint]
2. Any proper closed subset has empty interior [Consider interior(Z), $X \setminus Z$]
3. Any non-empty open is dense [$U \subseteq$ interior$(\overline{U})$]
4. If $X = Y \cup Z$ for $Y, Z \subseteq X$ closed, then $Y = X$ or $Z = X$

Def [Vak 3.6.4] Say a topological space is *irreducible* if the conditions above hold. A scheme is so if its topological space is.

Absence of bump functions:



$\rightsquigarrow$ Recall this can't happen in complex analysis

Corollary (of Corollary). If $f \in \mathcal{O}_X(X)$ vanishes on an open set then $f = 0$.

Proof $\mathbb{V}(f) := \{ \mathfrak{p} \in X \mid f \in \mathfrak{m}_{\mathfrak{p}} \}$ is closed, so $\mathbb{V}(f) = X$. Thus $\forall U \in X$ affine,

$$f|_U \in \bigcap_{\substack{\mathfrak{p} \subseteq \mathcal{O}_X(U) \\ \text{prime}}} \mathfrak{p} =: \text{nilradical}(\mathcal{O}_X(U)) = 0 \quad \because \mathcal{O}_X(U) \text{ is a domain.}$$

Thus $f = 0$ by sheaf axioms. $\square$

Example [Vak 5.2.G] Spec A is integral $\Leftrightarrow$ A is a domain

Proof ($\Leftarrow$) For $U \subseteq \operatorname{Spec} A$ open, take cover $U = \bigcup_i U_i$ by distinguished opens $U_i = \operatorname{Spec} A[\frac{1}{a_i}]$

Suppose $f, g \in \mathcal{O}_X(U)$, $fg = 0$. Then $\forall i$, $f|_{U_i} \cdot g|_{U_i} = 0$ in $A[\frac{1}{a_i}] \rightsquigarrow$ loc of a domain

Hence $\forall i$, $f|_{U_i} = 0$ or $g|_{U_i} = 0$ $\Rightarrow \mathcal{O}(U_i)$ is a domain

If $g|_{U_i} = 0$ $\forall i$ then $g = 0$ by sheaf axioms.

Suppose $g|_{U_0} \neq 0$, then $f|_{U_0} = 0$. Then $\forall i$, $U_i \cap U_0$ is non-empty and $f|_{U_i}$ vanishes on $U_i \cap U_0$ so $f|_{U_i} = 0$ $\forall i$ so $f = 0$. $\square$

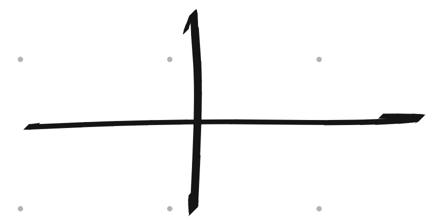
Examples: $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$, $\qquad \operatorname{Spec} k[x_1^{\pm}, \dots, x_n^{\pm}]$

Open subschemes of $\mathbb{A}^n_k$ ($\mathbb{G}_m$, tori, $\mathbb{A}^2 \setminus \underline{0}$)

$\operatorname{Spec} k[x, x^{-1}]$

$\mathbb{P}^n_k$ [Hint: same argument but with standard affine cover]

Non-examples: two things can go wrong —



① Zero divisors eg $k[x,y]/(xy)$ $\rightsquigarrow$ topological, global

② Nilpotents eg $k[x]/(x^2)$ $\rightsquigarrow$ algebraic, local

Def [Vak 5.2.1] A ring is reduced if it has no nilpotents, ie $\sqrt{(0)} = (0)$

A scheme X is reduced if $\mathcal{O}_X$ is a sheaf of reduced rings.

★ Theorem [Vak 5.2.F] Reduced & irreducible $\Leftrightarrow$ Integral.

Proof. ($\Rightarrow$) Suppose X is irreducible but not integral, ie $\exists U \subseteq X$ open with $f, g \in \mathcal{O}_X(U)$, $f, g \neq 0$, $fg = 0$.

So $U = \mathbb{V}(fg) = \mathbb{V}(f) \cup \mathbb{V}(g)$

U is open in irreducible so U is irreducible

Wlog $U = \mathbb{V}(f)$.

Choose affine open $\operatorname{Spec} A \subseteq U$, $f|_{\operatorname{Spec} A}$ vanishes everywhere on $\operatorname{Spec} A$ so $f|_A$ nilpotent. $\square$

Checking for reducedness.

Sanity [Vak 5.2.B] $X = \operatorname{Spec} A$ is reduced iff A is.

Proof. If $U \subseteq \operatorname{Spec} A$ is such that $\exists f \in \mathcal{O}_X(U)$, $f^n = 0$, $f \neq 0$ then wlog U is a distinguished open $\operatorname{Spec} A[\frac{1}{a}]$. Then $f = \bar{f}/a^\ell$ $(\bar{f} \in A)$ is such that $(\bar{f}/a^\ell)^n = 0$ i.e. $\exists m$, $a^m \bar{f}^n = 0$. But then $a\bar{f} \in A$ is nilpotent and non-zero ($\because f \neq 0$) $\square$

Recall commie alg fact: the natural map $A \xrightarrow{\varphi} \prod_{\mathfrak{p} \in \operatorname{Spec} A} A_{\mathfrak{p}}$ is injective [If $f \neq 0$ then $\operatorname{Ann}(f)$ is a proper ideal, choose $\mathfrak{p}$ maximal containing $\operatorname{Ann}(f)$. Then $f/1 \neq 0$ in $A_{\mathfrak{p}}$]

Follows that $\operatorname{Spec} A$ is reduced $\iff$ Stalks $A_{\mathfrak{p}}$ are reduced $\forall$ primes $\mathfrak{p}$

$\iff$ Stalks $A_{\mathfrak{p}}$ are reduced $\forall$ maximals $\mathfrak{p}$.

Compact = $\underbrace{QC}_{\text{open cover fin subcover}}$ + QS

$\underbrace{}$ has finite cover by affines, $\cap$ of any two affines is finitely covered by affines

Reducedness is

① Stalk-local, i.e. suffices to check all stalks.

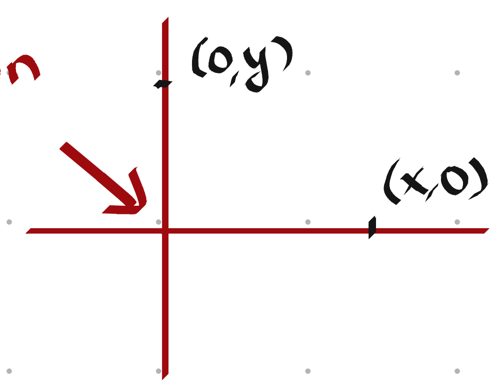
② Affine local, ie. suffices to check on an affine cover

③ Can be checked on closed points if X is quasicompact.

④ !! Not open, for example $k[x,y]/(x^2,xy) \ni f(y) + \lambda x = g$

$\Rightarrow$ g is determined by $g(0,y)$ and $\frac{\partial g}{\partial x}(0,0)$

stalk at origin is reduced



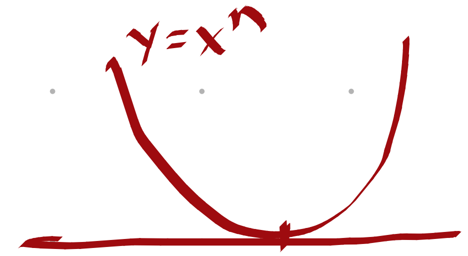
only nonreduced stalk

"horizontal fuzz"

$V(x^2) \subseteq \operatorname{Spec} k[x,y]/(xy)$

$\left(\frac{k[x,y]}{(y^2,x)}\right)_{(x,y)} \cong k[y,y^{-1}]$ $\operatorname{Spec} k[x,y]/(x^2,xy)$

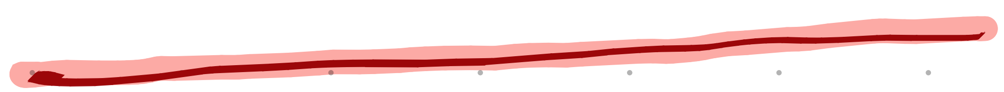
Other examples: $k[x]/(x^n) \cong k[x,y]/(y-x^n, y)$



fuzz of degree n

$\dfrac{k[x,y]}{(y-x^2, y-t)}$

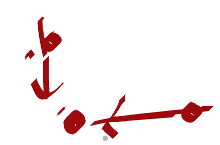
$k[x,y]/(x^2) \cong k[x]/(x^2) \otimes_k k[y]$

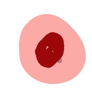


y axis fattened

$\dfrac{k[x]}{(x^3)}$

$k[x,y]/(x^2, xy, y^2)$





If A is a ring with nilradical $\mathfrak{n}$, then $A_{red} = A/\mathfrak{n}$ is reduced and the map $A \twoheadrightarrow A/\mathfrak{n}$ induces a map $\operatorname{Spec} A_{red} \hookrightarrow \operatorname{Spec} A$.

$$\operatorname{Spec} A_{red} = V(\mathfrak{n}) = \overline{\{\eta\}}$$

Check: this is a homeomorphism of topological spaces

Check: if B is a reduced ring, $\varphi: A \to B$ a ring map then $\mathfrak{n} \subseteq \ker \varphi$

Check: doing this on an affine cover, the local constructions glue.

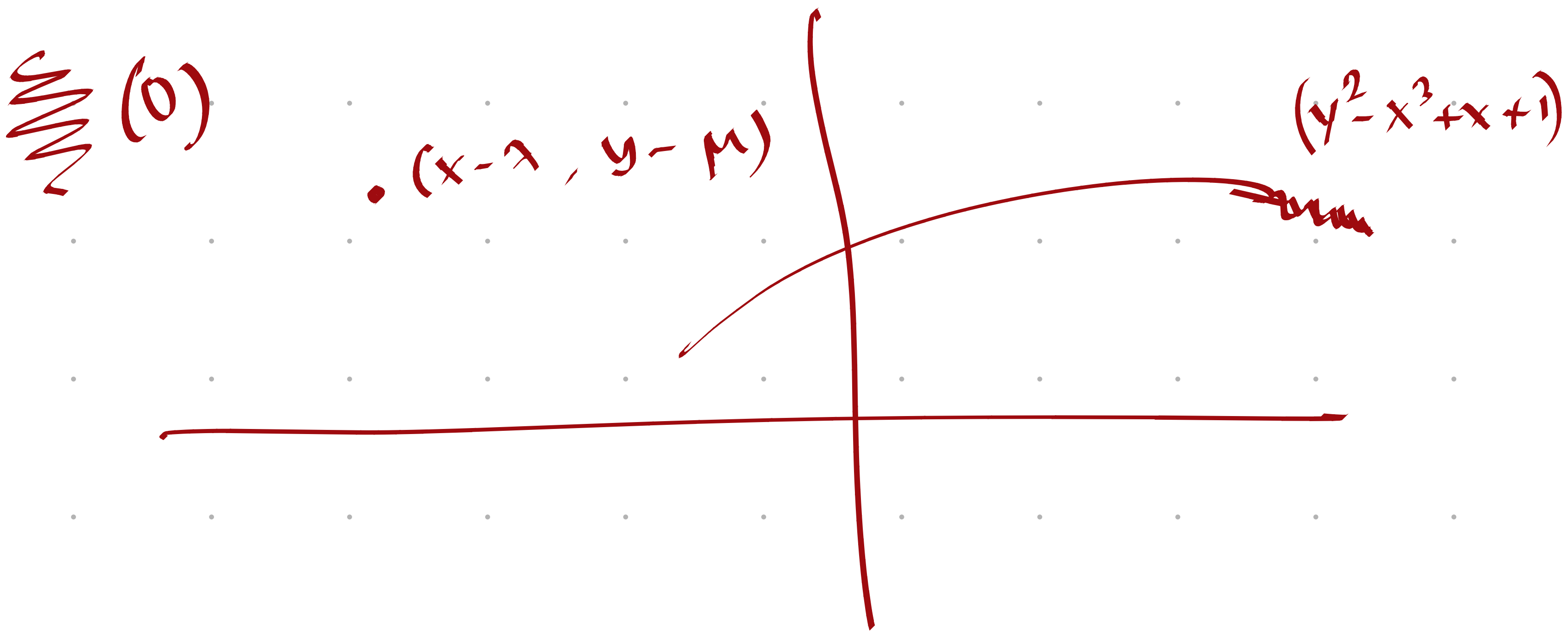
The reduced structure of a scheme X is a morphism $X_{red} \to X$ (reduced) with the universal property that $\forall$ scheme maps $Y \to X$ with Y reduced,

$$\begin{tikzcd} & Y \arrow[dl, dashed, "\exists!"'] \arrow[d] \\ X_{red} \arrow[r, hook] & X \end{tikzcd}$$

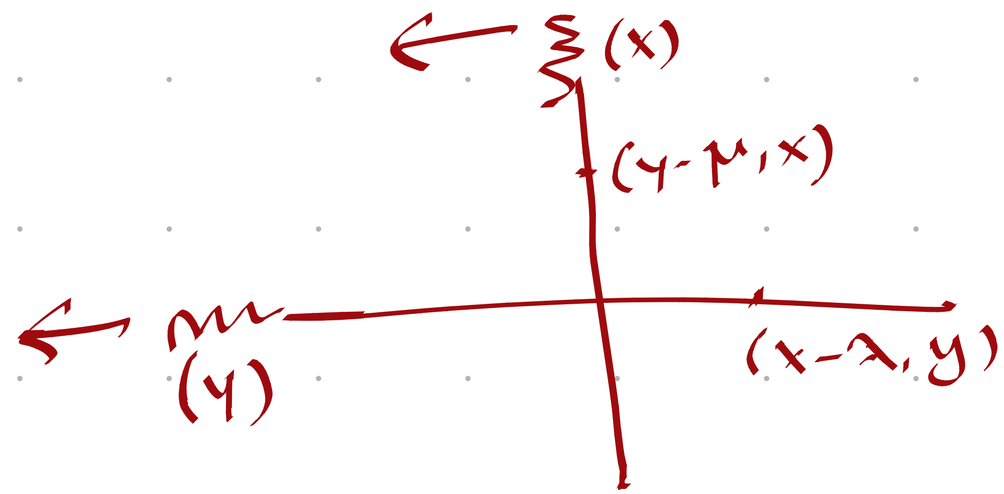
Generic and associated points.

$(k=\bar{k}$, char $k=0)$

Recall picture of Spec $k[x]$, Spec $k[x,y]$.



$$\operatorname{Spec} \frac{k[x,y]}{(x,y)} = X\text{axis} \cup Y\text{axis}$$



Irreducible schemes have "generic points":

If $\operatorname{Spec} A$ is irreducible, then A has a unique minimal prime ideal $\mathfrak{p}$.

Proof. The nilradical is prime: if fg is nilpotent, then $\operatorname{Spec} A = \mathbb{V}(fg) = \mathbb{V}(f) \cup \mathbb{V}(g)$ wlog $\operatorname{Spec} A = \mathbb{V}(f)$ so f nilpotent. $\square$

If X is irreducible, then choose an affine open $U \subseteq X$. U is irreducible so has a generic point η. Say η is the generic point of X.

This is well-defined:

$$\begin{tikzcd}
 & \operatorname{Spec} C \arrow[dl] \arrow[dr] & \\
\underline{\operatorname{Spec} A} \arrow[dr] & & \operatorname{Spec} B \arrow[dl] \\
 & X &
\end{tikzcd}$$

(nilradicals pull back)

Example. A function is non-zero $\iff$ it is $\overbrace{\text{supported at the generic point}}^{f \in \mathcal{O}_\eta \text{ is non-zero}}$

Observe that $\overline{\{\eta\}}$ is precisely X_{red}.

Intuition: minimal primes should capture the reduced geometry

[Why? If $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ in A are minimal then nilradical is

$\mathfrak{N} = \bigcap_{i=1}^{n} \mathfrak{p}_n$ so $\mathbb{V}(\mathfrak{p}_1, \ldots, \mathfrak{p}_n) = \overline{\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}} = \mathbb{V}(\mathfrak{N})$]

Observe [Vak 5.5.12] everything in a minimal prime is a zero divisor.

If $\mathfrak{p} \subseteq A$ is minimal, $f \in \mathfrak{p}$ then $\mathfrak{p}A_\mathfrak{p}$ is the unique prime in $A_\mathfrak{p}$, so $f/1$ vanishes at all points, so $f/1$ is nilpotent in $A_\mathfrak{p}$. So $\exists g \in A \setminus \mathfrak{p}$ $gf^n = 0$

- Notions of support, vanishing, etc

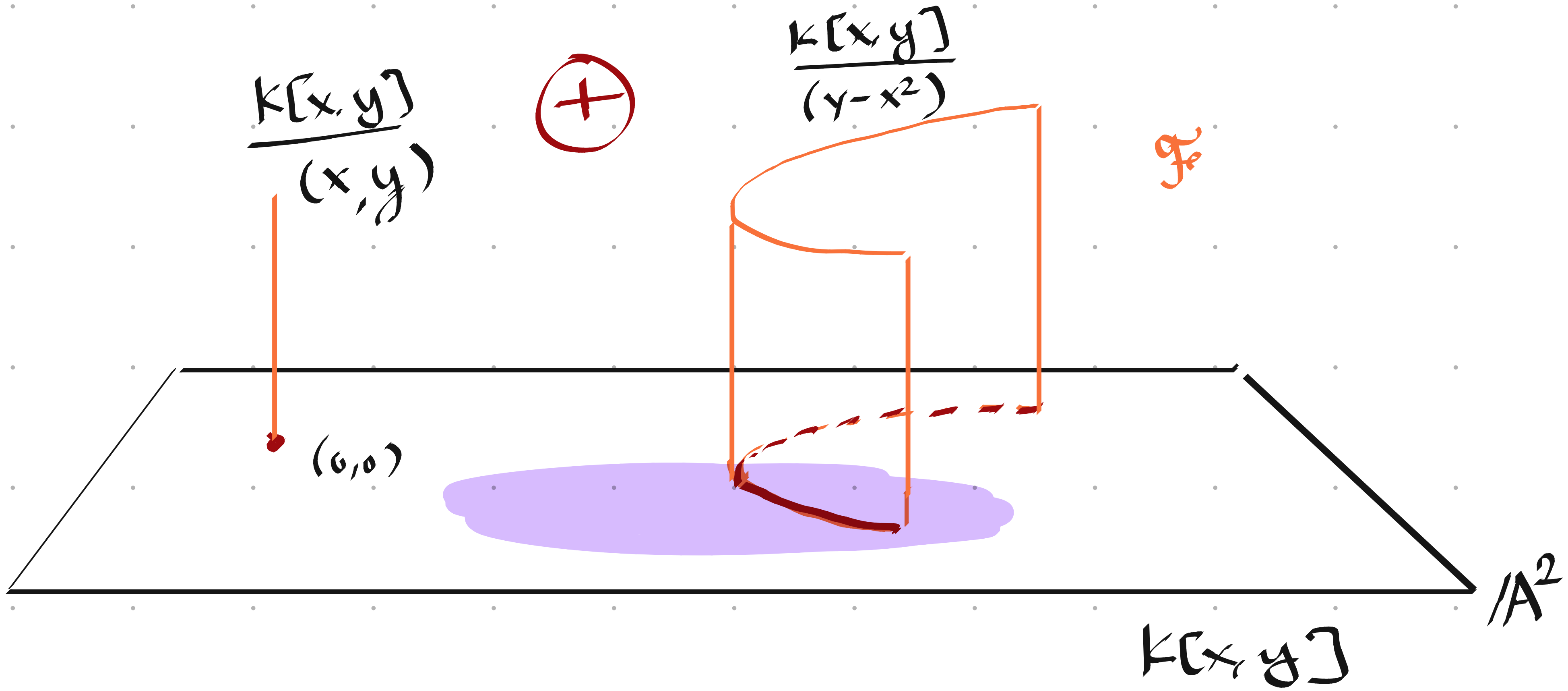
Let M be an A-module.

$\forall x \in A \setminus \mathfrak{p},\ xm \neq 0$

$\Leftrightarrow \mathrm{Ann}(m) \subseteq \mathfrak{p}$

For $m \in M$, say $\mathrm{Supp}(m) = \left\{ \mathfrak{p} \in \mathrm{Spec}\, A \;\middle|\; \frac{m}{1} \neq 0 \text{ in } M_{\mathfrak{p}} \right\} = \mathbb{V}(\mathrm{Ann}(m))$

$$\mathrm{Supp}(M) = \bigcup_{m \in M} \mathrm{Supp}(m) = \{ \mathfrak{p} \in \mathrm{Spec}\, A \mid M_{\mathfrak{p}} \neq 0 \} = \mathbb{V}(\mathrm{Ann}(M))$$



Def [Vak 5.5.8] $\mathfrak{p} \in \mathrm{Ass}(M) \iff \exists\, m \in M,\ \mathfrak{p} = \mathrm{Ann}(m)$

$\iff$ $\exists$ an injection $A/\mathfrak{p} \longrightarrow M$

[$(\Rightarrow)$ $A/\mathfrak{p} \cong A \cdot m \subseteq M$

$(\Leftarrow)$ choose $1 \in A/\mathfrak{p} \hookrightarrow M$]

Easy check: $\mathrm{Ass}_{S^{-1}A}(S^{-1}M) = \mathrm{Ass}_A M \cap \mathrm{Spec}\, S^{-1}A$. ("ass is stalk-local")

Theorem. ① $\mathrm{Ass}(M) \subseteq \mathrm{Supp}(M)$

② $\mathrm{Ass}(M) = \emptyset \iff M = 0$

③ $\mathrm{minimal}(\mathrm{Supp}\, M) \subseteq \mathrm{Ass}(M)$

Sketch. ① If $\mathfrak{p} \in \mathrm{Ass}(M)$ then $A/\mathfrak{p} \subseteq M$, and $(A/\mathfrak{p})_{\mathfrak{p}} \neq 0$

③ idea: if $\mathfrak{p} \subseteq A$ is minimal in Supp then $A_{\mathfrak{p}}$ has just one point and $M_{\mathfrak{p}} \neq 0$ so $\mathrm{supp}(M_{\mathfrak{p}})$ must contain that point by ②

Associated points see

Zero-ness [Vak 5.5.D] $\overline{\{\mathfrak{p} \in \operatorname{Ass} M \mid m \neq 0 \text{ in } M_{\mathfrak{p}}\}} = \operatorname{Supp} \underline{m}$.

Nilpotency [Vak 5.5.E] $\overline{\{\mathfrak{p} \in \operatorname{Ass} A \mid A_{\mathfrak{p}} \text{ non-reduced}\}}$

$= \{\mathfrak{p} \in \operatorname{Spec} A \mid A_{\mathfrak{p}} \text{ non-reduced}\}$

Zero divisors. $f \mid 0 \Leftrightarrow \exists\, \mathfrak{p} \in \operatorname{Ass}(A),\ \mathfrak{p} \in \mathbb{V}(f)$.

"weakly associated"

(A) $\mathrm{Ass}(M) = \bigcup_{m \in M} \text{minimal}(\mathrm{Supp}(m))$. ("ass-primes are weak-ass")

Proof. ($\Rightarrow$) If $\mathfrak{p} = \mathrm{Ann}(m)$ then $\mathfrak{p}$ is minimal in $\mathrm{Supp}(m) = \mathbb{V}(\mathrm{Ann}(m)) = \mathbb{V}(\mathfrak{p})$

($\Leftarrow$) For $m \in M$, $\mathfrak{p} \in \mathrm{Supp}(m)$ minimal, note $N = A \cdot m \cong A/\mathrm{Ann}(m) \subseteq M$

$\mathfrak{p} \in \text{minimal}(\underbrace{\mathrm{Supp}\, A/\mathrm{Ann}\, m}_{\mathbb{V}(\mathrm{Ann}(m))}) \subseteq \mathrm{Ass}(A/\mathrm{Ann}\, M) \subseteq \mathrm{Ass}\, M$. $\square$

Example. Take $M = A$, $m = 1$, then $\text{minimal}(\mathrm{supp}(1))$ = minimal primes of $\mathrm{Spec}\, A$. So you're recovering reduced geometry of $\mathrm{spec}\, A$.

Say $\mathfrak{p}$ is embedded if it is not minimal.

<u>Fact</u> $M \longrightarrow \prod_{\mathfrak{p} \in \mathrm{Ass}\, M} M_{\mathfrak{p}}$ is an injection

(B) If A is Noetherian, M f.g then $\mathrm{Ass}(M)$ is finite.

Sketch. Build filtration $0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots \subseteq M_n = M$ with $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for some prime. [Vak 5.5.M]

$M \neq 0 \Rightarrow \mathrm{Ass}\, M \neq \emptyset$, pick $\mathfrak{p} \in \mathrm{Ass}\, M$, set $M_1 = A/\mathfrak{p}_1 \hookrightarrow M$ by thm. Then repeat with M/M_1. $\square$

Observe if $N \subseteq M$, then $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(N) \cup \mathrm{Ass}(M/N)$. [Vak 5.5.L]

???

$$\mathrm{Ass}\, M \subseteq \underbrace{\mathrm{Ass}(M_0)}_{\{\mathfrak{p}_0\}} \cup \mathrm{Ass}\underbrace{(M_1/M_0)}_{\{\mathfrak{p}_1\}} \cup \dots \cup \mathrm{Ass}\underbrace{(M_n/M_{n-1})}_{\{\mathfrak{p}_n\}}$$

Profit. $\square$

(c) $\text{Zerodivisors}(M) := \{a \in A \mid \exists m \in M \setminus 0,\ a \cdot m = 0\} = \bigcup_{p \in \operatorname{Ass} M} p$

Sketch. ($\Leftarrow$) Obvious.
($\Rightarrow$) Follows from

Lemma. $\text{maximal}(\operatorname{Ann}(m) \mid m \in M) \subseteq \operatorname{Ass} M$

"maximal things in sets of proper ideals have a tendency to be prime" covered by noeth rings

Non-affine case: if X locally Noetherian, then

noeth as sch $\Leftrightarrow$ loc noeth + top noeth

$p \in \mathrm{Ass}(X) \Leftrightarrow \forall\, U \ni p$ affine open, $p \in \mathrm{Ass}(U)$.

A rational function is an element of $\mathcal{O}_X(U)$ such that $\mathrm{Ass}(X) \subseteq U$, up to obvious compatibility.

Special case: Integral schemes have a unique associated prime, the generic point η. The field of functions is $\mathcal{O}_{X,\eta}$

(0)

$\forall U$ open, $\mathcal{O}_X(U) \hookrightarrow \mathcal{O}_{X,\eta}$

intersection of affine opens
can be covered by affines that
are simultaneously distinguished

Reduced $\iff$ all stalks reduced.

Normal $\iff$ all stalks are (normal domains) = int closed in field of fractions

factorial $\iff$ all stalks are UFDs