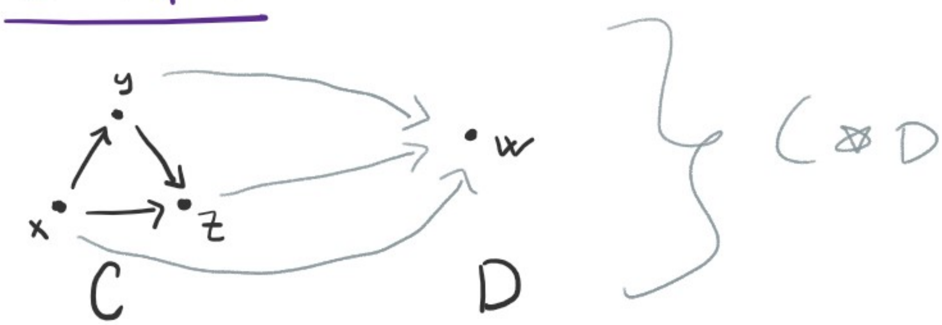
Joins & Slices

In ordinary category theory ...

Def Given two categories C & D we define the **join $C \star D$** by

- objects are $ob(C) \sqcup ob(D)$
- morphisms $Hom(x,y)$ are
  - $Hom_C(x,y)$ if $x,y \in C$
  - $Hom_D(x,y)$ if $x,y \in D$
  - $*$ if $x \in C$, $y \in D$
  - $\varnothing$ if $x \in D$, $y \in C$

Example



Def Given an object $x \in C$ of a category, the **slice categories** are the **over category** $C_{x/}$ and **under category** $C_{/x}$ of maps out of (resp. into) x. Morphisms are **commutative triangles** i.e.

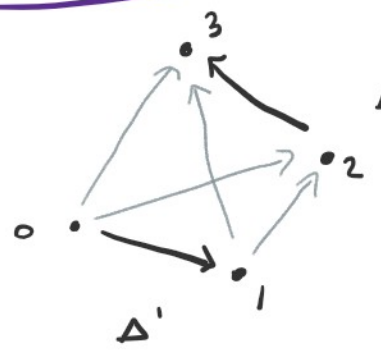
$$\begin{tikzcd} x \arrow[r, "g"] \arrow[d, "f"'] & z \\ y \arrow[ur, "f \to g"'] & \end{tikzcd}$$

In simplicial sets

In simplicial sets

Lets try to define join internal to sSet.

Example What should $\Delta^1 \star \Delta^1$ be? Or more generally $\Delta^n \star \Delta^m$?

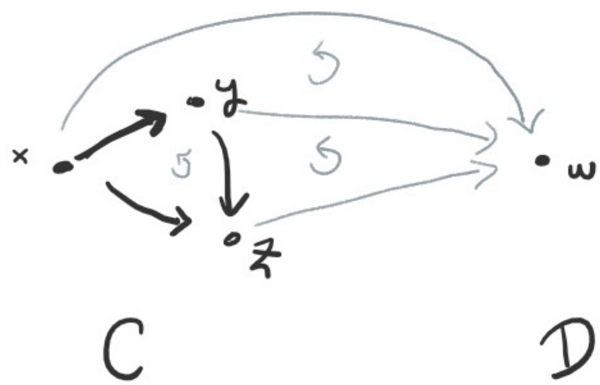


$\Delta^1 \star \Delta^1 = \Delta^3$

So $\Delta^n \star \Delta^m = \boxed{\Delta^{n+m+1}}$

Example

What should $N(C) \star N(D)$ be?



So $N(C) \star N(D) = \boxed{N(C \star D)}$

Informal definition

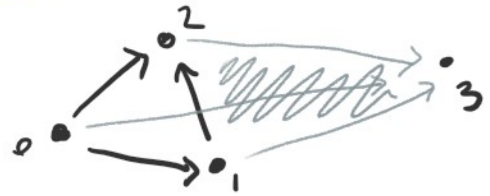
$X \star Y$ has simplices the disjoint union of simplices of X and Y, plus

- A unique 1-simplex $x < y$ for each $x \in X_0$ and $y \in Y_0$
- Unique n-simplices "inbetween X and Y" whenever the "relevant faces" exist in X and Y

whenever the "relevant faces" exist in X and Y

The face and degeneracy maps are "obvious"

Example



 are 2-simplices

 is a 3-simplex iff  is a 2-simplex in X

Def (formal) For $X, Y \in sSet$, define $X \star Y$ by

$$(X \star Y)_n = \coprod_{([a],[b]) \in Cut([n])} X_a \times Y_b$$

$$|\underbrace{0<1<2}_{[3]}|\underbrace{3<\dots<n}_{[n-2]}|$$

(convention: $X_\emptyset = *$)

Given a map $\alpha : [n] \to [m]$
and a cut $([a'],[b'])$ of $[m]$
α uniquely restricts to $\alpha_1 : [a] \to [a']$
$\alpha_2 : [b] \to [b']$
where $([a],[b])$ is a cut of $[m]$.

So $(X \star Y)(\alpha) : (X \star Y)_m \to (X \star Y)_n$
is defined by $X_{a'} \times Y_{b'} \xrightarrow{(X(\alpha_1), Y(\alpha_2))} X_a \times Y_b$.

Example

Face and degeneracy maps:

Face and degeneracy maps:

$d_0(0<1,2) = \boxed{(1,2)}$ $\quad s_0(0<1,2) = \boxed{(0<0<1,2)}$

$d_1(0<1,2) = \boxed{(0,2)}$ $\quad s_1(0<1,2) = \boxed{(0<1<1,2)}$

$d_2(0<1,2) = \boxed{(0<1,\emptyset)}$ $\quad s_2(0<1,2) = \boxed{(0<1,2<2)}$

Proposition 1.4.12

For $X, Y \in s\mathcal{S}et$,

$X \star Y$ is an ∞-cat $\iff$ X & Y are ∞-cats

Proof sketch

$(\Rightarrow)$

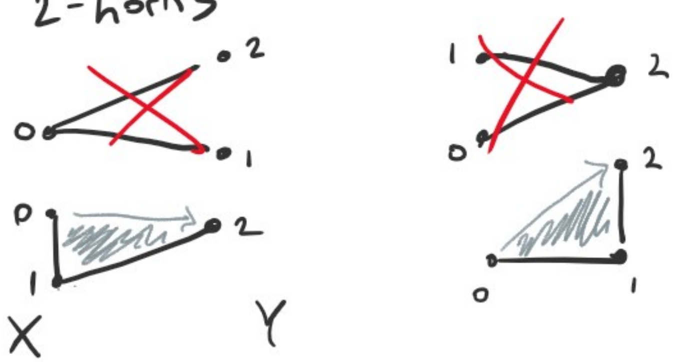
$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \star Y \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

also true for diagrams "contained in" X or $Y \subseteq X \star Y$

$(\Leftarrow)$

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \star Y \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

inner 2-horns



Slices

Note that $S \star -$ determines a functor

$S \star - : s\mathcal{S}et \to s\mathcal{S}et_{S/}$, $\quad S \star X \mapsto S \hookrightarrow S \star X$

and similarly

$-\star S : sSet \to sSet_{S/}$.

Let's try to build a right adjoint!

Familiar strategy

(1) Let $p : S \to X$.

Then define $X_{p/}$ by

$$\mathrm{Hom}_{sSet}(\Delta^n, X_{p/}) := \mathrm{Hom}_{sSet_{S/}}(S \star \Delta^n, X)$$

so the adjunction holds on representables

(2) $S \star -$ preserves colimits.

Pf: category theory

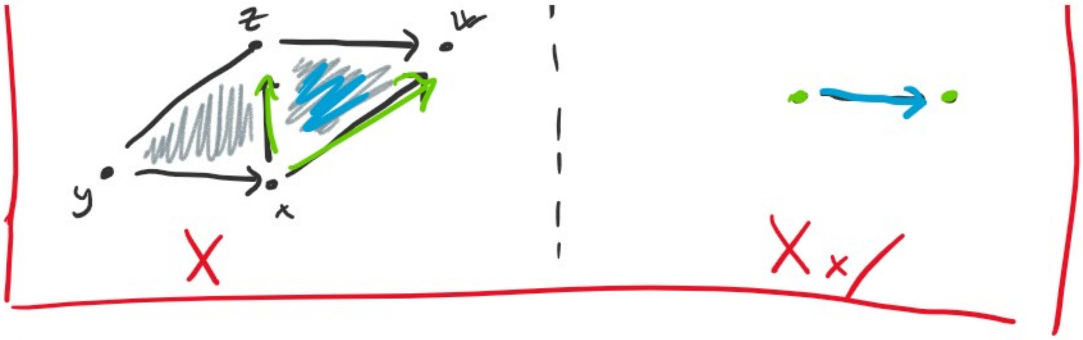
(3) $X_{p/}$ is a right adjoint to $S \star -$

(similarly, $X_{/p}$ is defined to be right adjoint to $- \star S$)

Example

Interpret an object $x \in X \in sSet$ as a map $x : \Delta^0 \to X$. What is $X_{x/}$?

Intuition



$$(X_{x/})_0 = \mathrm{Hom}_{x/}(\underset{\Delta^1}{\underset{\shortparallel}{\Delta^0 * \Delta^0}}, X)$$

$= \{ 1 \text{ simplices out of } x \}$

$$(X_{x/})_1 = \mathrm{Hom}_{x/}(\underset{\Delta^2}{\underset{\shortparallel}{\Delta^0 * \Delta^1}}, X)$$

Remark

A cone over a sSet K is

$\Delta^0 * K$

this will later let us define colimit.

Def For $\mathcal{C} \in \mathcal{C}at$, define the twisted arrow category $Tw(\mathcal{C})$ as

- $ob(Tw(\mathcal{C})) = Mor(\mathcal{C})$

• $\mathrm{Mor}(\mathrm{Tw}(\mathcal{C})) =$

$$\begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "\alpha"'] & y' \\ x \arrow[r, "f"] & y \arrow[u, "\beta"'] \end{tikzcd} \qquad (\alpha, \beta) : f \to f'$$

Lemma

This defines a functor

$$\mathrm{Tw}(s\mathcal{S}et) \longrightarrow s\mathcal{S}et$$

$$p : S \to X \longmapsto X_{p/}$$

Pf $(\varphi \to f\varphi i) \longmapsto (X_{\varphi/} \to Y_{f\varphi i/})$

Strategy

(i) $X_{\varphi/} \to X_{\varphi i/}$ $\rightsquigarrow$ corresponds to

$$\begin{tikzcd} A \arrow[r, "\varphi i"] \arrow[d, "i"'] & X \arrow[d, equal] \\ B \arrow[r, "\varphi"] & X \end{tikzcd}$$

$X_{\varphi/} \to Y_{f\varphi/}$ $\rightsquigarrow$

$$\begin{tikzcd} B \arrow[r, "f\varphi"] \arrow[d, equal] & Y \\ B \arrow[r, "\varphi"] & X \arrow[u, "f"'] \end{tikzcd}$$

writing A map $X_{\varphi/} \to X_{\varphi i/}$ is the same as writing

$A \star X_{\varphi/} \to X$ under A by adjunction.

Choose $A \star X_{\varphi/} \xrightarrow{i \star id} B \star X_{\varphi/} \xrightarrow{\text{counit}} X$

Similarly,

writing a map $X_{\varphi/} \longrightarrow Y_{f\varphi/}$ is the same as writing a map $X_{\varphi/} \longrightarrow Y_{f\varphi/}$ is the same as writing $B \star X_{\varphi/} \to Y$. Choose

$$\beta \star X_{\varphi/} \xrightarrow{\text{counit}} X \xrightarrow{f} Y$$

get

$$\begin{tikzcd} X_{\varphi/} \arrow[r] \arrow[d] & X_{\varphi i/} \arrow[d] \\ Y_{f\varphi/} \arrow[r] & Y_{f\varphi i/} \end{tikzcd}$$

by same construction

corresponds to

$$\begin{tikzcd} A \arrow[r, "\varphi i"] \arrow[d, "i"'] & X \arrow[d, equal] \\ B \arrow[r, "\varphi"] & X \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[r, "f\varphi"] \arrow[d, equal] & Y \\ B \arrow[r, "\varphi"] & X \arrow[u, "f"'] \end{tikzcd}$$

Need to show this commutes.

adj $B \star X_{\varphi/} \xrightarrow{\text{counit}} X \xrightarrow{f} Y$

adj $A \star Y_{f\varphi/} \xrightarrow{i \star id} B \star Y_{f\varphi/} \xrightarrow{\text{counit}} Y$

the composite is adj to a map $A \star X_{\varphi/} \longrightarrow Y$ ... (?)

Lemma the following lifting problems are equivalent

$$\begin{tikzcd} S \arrow[r] \arrow[d] & X_{\varphi/} \arrow[d] \\ T \arrow[r] \arrow[ur, dotted] & Y_{f\varphi i/} \end{tikzcd} \iff \begin{tikzcd} B \arrow[r] \arrow[rr, bend left, "\varphi"] & A \star T \sqcup_{A \star S} B \star S \arrow[r] \arrow[d] & X \arrow[d] \\ & B \star T \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

Pf Unpack everything?

Lemma

If $i : A \to B$ & $g : S \to T$ are monos,

$$i \hat{\star} g : A \star T \coprod_{A \star S} B \star S \longrightarrow B \star T$$

is a mono &

(1) I.A. if $\underline{i \text{ is RA } \text{or } g}$ is LA

(2) LA if i is LA

(3) RA if g is RA

Pf Consider the set $\{$monos $i : A \to B$ s.t. $i \hat{\star} g$ is I.A. for all monos $g\}$.

i. Show it is saturated

ii. Show it $\underline{\text{contains}}$ $\Lambda^n_i \to \Delta^n_i$ $0 < i \le n$

there $\{\Lambda^n_i \to \Delta^n_i\} \subset$ original set

$$\begin{gathered} \| \\ \left\{ \begin{matrix} \text{RA } i : A \to B \\ \text{s.t. } -\|- \end{matrix} \right\} \end{gathered}$$

i. it is saturated because it is the set of all monos with the LLP wrt

$$X_{\varphi/} \longrightarrow X_{\varphi i} \times_{Y_{f\varphi i}} Y_{f\varphi/}$$

for $f : X \to Y$ IF (?)